Digital Communication Systems Signal Detection, Error Probability, and Modulation

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Analogy between vector and signal

There is a complete analogy between the signal and the vector. In the N-dimensional Euclidian space we may define a vector with lengths and angles.

$V = Ax + By + Cz$, when all the three

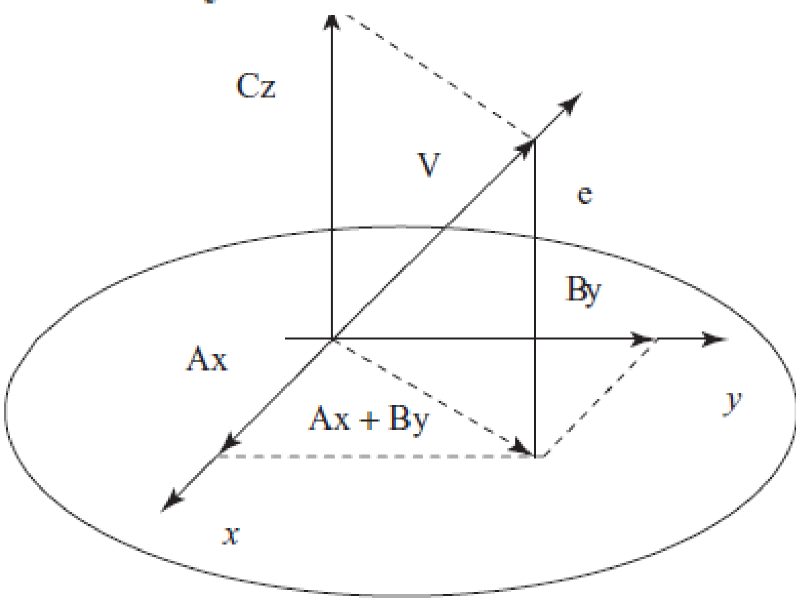


Fig. 5.1 *Three-dimensional representation of a vector*

Coordinates are mutually orthogonal to each other error **e in the approximation is zero when V is approximated in terms of three mutually orthogonal vectors x, y and z.**

The three vectors **x, y and z represent the complete set of orthogonal vectors in the three-dimensional space.**

Such vectors are known as *basis vectors. If a set of vectors {xi} is not complete, the error vector would not be zero in the approximation process.*

The choice of basis vectors is not unique, but set of basis vectors corresponds to a particular choice of coordinate system. Summarizing, we can say, if a set of vectors $\{\mathbf{x}_i\}$ is mutually orthogonal, if

$$x_m . x_n = \begin{cases} 0 & \text{for } m \neq n \\ |x_m|^2 & \text{for } m = n \end{cases} \tag{5.1}$$

vector $V = c_1\mathbf{x} + c_2\,\mathbf{y} + c_3\,\mathbf{z}$, where c_i (i = 1,2,3) is the constant

$$c_i = \mathbf{V}.\mathbf{x}_i \,/\, \mathbf{x}_i \,.\, \mathbf{x}_i = \mathbf{V}.\,\mathbf{x}_i \,/\, |\mathbf{x}_i|^2, \qquad i = 1, 2, 3, \ldots, N$$

Orthogonal signal space

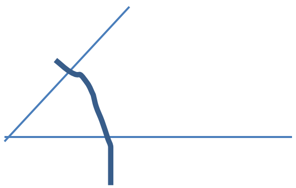
Let orthogonality of a signal set $\{s_i = (s_1, s_2, s_3, \ldots s_N)\}$ ***over the time interval [t1, t2] is defined as***

$$\int_{t_1}^{t_2} s_m(t).s_n^*(t)\,dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

Orthogonal signal space

If En = 1 for all n, the set is normalized and is called orthonormal set. The orthogonal set can always be normalized by dividing $s_n(t)$ by $\sqrt{E_N}$ for all n. So, the signals can be approximated over the interval [t1, t2] by a set of mutually orthogonal signal $s_1(t)$, $s_2(t)$, ...$s_N(t)$ as,

$$s(t) = \sum_{n=1}^{N} c_n s_n(t) \quad t_1 \le t \le t_2$$



when the set is complete, the error energy E_e *goes to zero and the perfect equality holds. The correlation coefficients c_n (= cos θ) is expressed as ,*

$$c_n = \cos\theta = \frac{\int_{t_1}^{t_2} s(t) s_n^*(t)\, dt}{\int_{t_1}^{t_2} s_n^2(t)\, dt} = 1/E_n \int_{t_1}^{t_2} s(t)\, s_n^*(t)\, dt$$

where *n = 1,2, …N and * represents the complex conjugate. Thus the value of c_n lies within [−1, 1]. If the two signals are orthogonal, the coefficient is 0, for maximum similarity, it is 1 and for maximum dissimilarity it is –1.*

When the set $\{s_n(t)\}$ is such that the error energy $Ee \to 0$ as $N \to \infty$ for every member of some particular class, then we say that set $\{s_n(t)\}$ is complete over [t1,t2] for that class s(t) and the set $\{s_n(t)\}$ is called the basis functions or basis signals.

$$c_n = \frac{1}{\sqrt{E_1\ E_2}} \int_{t_1}^{t_2} s_1(t)\ s_2^{\,*}(t)\ dt$$

Geometric Representation of Transmitted Signals

Geometric representation of the transmitted signal is a very useful method to take correct decision for a signal in presence of noise

we represent any set of *M energy signals $\{s_i(t)\}$ as linear combinations of N orthonormal basis functions,*

$$s_i(t) = \sum_{i=1}^{N} s_{ij}\ \varphi_j(t), \qquad N \leq M. \qquad \text{For } 0 \leq t \leq T \text{ and } i = 1,2,3, \ldots M$$

$$s_{ij} = \int_0^T s_i(t)\ \varphi_j(t)\ dt, \text{ for } i = 1, 2, \ldots M,\ j = 1, 2, \ldots N$$

s_{ij} is known as the coefficient of expansion. Now the real valued functions $\varphi_1(t), \varphi_2(t), \varphi_3(t), \ldots, \varphi_N(t)$ are orthonormal. The word orthonormal means that

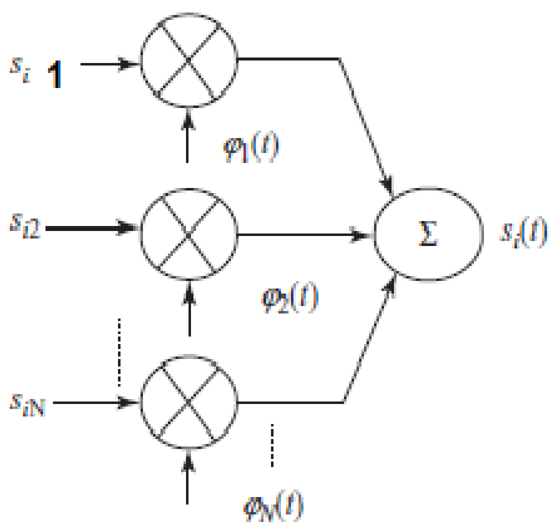
$$\int_0^T \varphi_i(t)\,\varphi_j(t)\,dt = \delta_{ij}, = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

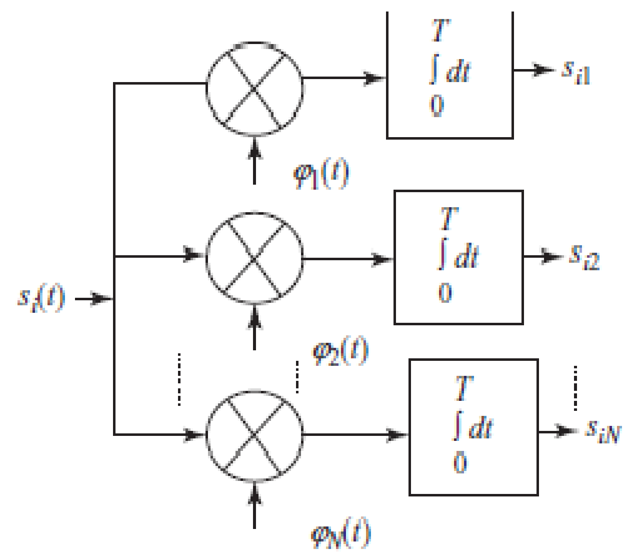
When $i = j$, the integral is 1, it implies that each basis function is normalized to have unit energy.

When $i \neq j$, the integral value is 0, it implies that the basis functions are orthogonal w.r.t each other over the interval $0 \leq t \leq T$.

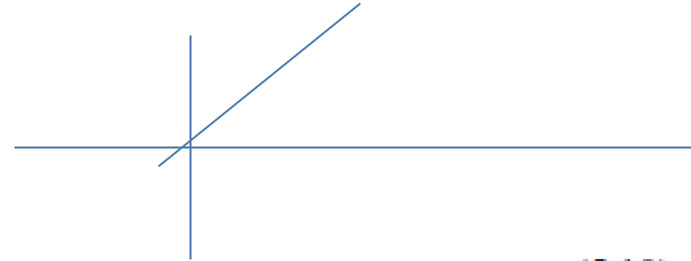
The coefficient s_{ij} is the j_{th} element of the N-dimensional vector space. The transmission system with input signal having M states (m_i = 1, 2, …M) and the modulated energy signal s_i (t) can be modeled as

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\,\varphi_j(t),$$





Each signal in the set $\{s_i(t)\}$ is completely defined by



$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, i = 1, 2, \dots M \tag{5.10}$$

where s_i is called the *signal vector*. The set of signal vectors may be defined in N-dimensional Euclidian spaces for M sets of points on $\{s_i\}$ with mutually perpendicular axes $\phi_1, \phi_2, \dots, \phi_N$.

The concept of correlator is very important in the receiver design, as at the receiver the first stage is the detection of the signal. The important assumption is that the basis function used in the product integrator of the receiver are in phase i.e., coherent with the basis functions used at the transmitting end. Hence, the reception is termed as *coherent reception.*

5.4 SCHWARZ INEQUALITY

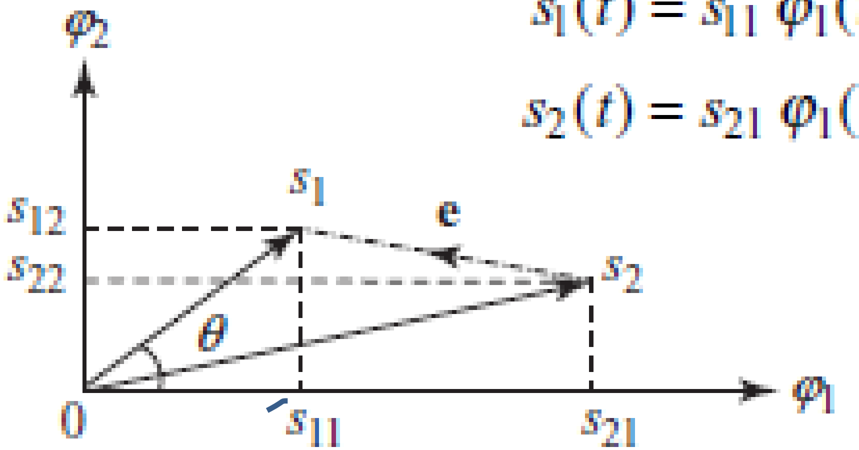
Considering two energy signals $s_1(t)$, and $s_2(t)$, the Schwarz inequality states that

$$\left(\int_{-\infty}^{\infty} s_1(t)\, s_2(t) dt\right)^2 \le \left(\int_{-\infty}^{\infty} s_1^{\,2}(t)\, dt\right)\left(\int_{-\infty}^{\infty} s_2^{\,2}(t)\right) \qquad E \le E_1 E_2$$

The equality holds if and only if *s1 (t) = c s_2 (t), where c is a constant.*

$$s_1(t) = s_{11}\, \varphi_1(t) + s_{12}\, \varphi_2(t)$$

$$s_2(t) = s_{21}\, \varphi_1(t) + s_{22}\, \varphi_2(t)$$



$$\cos\theta = \frac{\left(\int_{-\infty}^{\infty} s_1(t) s_2(t) dt\right)}{\left(\int_{-\infty}^{\infty} s_1^{\,2}(t)\, dt\right)^{1/2} \left(\int_{-\infty}^{\infty} s_2^{\,2}(t)\, dt\right)^{1/2}}$$

where $\varphi_1(t)$, $\varphi_2(t)$ have satisfied the orthonormality conditions over the entire time interval $(-\infty, \infty)$.

As $|\cos\theta| \le 1$, Schwarz equality follows

Signal Representation and Baseband Processing

A digital modulator is supposed to accept stream of information-bearing symbols (usually bits) and represent them appropriately with or without the help of a carrier.

So, a very important issue in-between is to represent information symbols in suitable energy signals so that the signals can be modulated, amplified and transmitted.
1100101000111, 1 has one symbol wave, o has other and orthogonal
For a continuous stream of input of information sequence what kind of strategy should we take to represent them as signals?
Let us consider a systematic approach to identify M symbols from the input information sequence. For example, if the information sequence is binary and if we choose M = 2, we can identify '1' as one symbol and '0' as the other.

Else, if we choose M = 4 for the same binary information sequence; we may consider a group of two bits at a time to define one symbol. 11, 00, 10, 01 for M=4, if 1 has bit duration T, 11 has 2T

The duration of a symbol now is twice the duration of one information bit. If the rate of incoming information is R_b bits/sec, the symbol rate is $R_b/2$ symbols per second. Usually, for practical considerations, M is so chosen that $M = 2^m$, where 'm' is a positive integer.

The next issue is to design 'M' energy signals for these M symbols such that the energy of each signal is limited within the symbol duration. This problem is addressed in general by a scheme known as Gram-Schmidt Orthogonalization

The principle of Gram-Schmidt Orthogonalization (GSO) states that, any set of M energy signals, $\{s_i(t)\}$, $1 \le i \le M$ can be expressed as linear combinations of N orthonormal basis functions, where $N \le M$.

GRAM-SCHMIDT ORTHOGONALIZATION PROCEDURE

At this point, for Gram-Schmidt Orthogonalization procedure, we require a complete orthonormal set of basis functions.

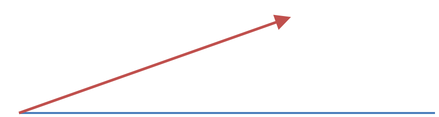
We need to prove that $\varphi_1(t)$, $\varphi_2(t)$, …, $\varphi_N(t)$ are orthonormal.

Let us consider that there are *M energy signals represented as* $s_1(t)$, $s_2(t)$ …, $s_M(t)$. *Let, the first basis function,*

$\varphi_1(t) = s_1(t)/\sqrt{E_1}$ where E_1 is the energy of the signal $s_1(t)$. Step 1

$s_1(t) = \varphi_1(t)\sqrt{E_1} = s_{11}\varphi_1(t)$, where coefficient $s_{11} = \sqrt{E_1}$ and $\varphi_1(t)$ has unit energy as required.

We define the coefficient $s_{21} = \int_0^T s_2(t)\,\varphi_1(t)\,dt$



A new intermediate function can be introduced as, Step 2

$$g_2(t) = s_2(t) - s_{21}\varphi_1(t)$$

which is orthogonal to $\varphi_1(t)$ over the interval [0,T], and the basis function $\varphi_1(t)$ has unit energy.

$$\varphi_2(t) = g_2(t) \Big/ \sqrt{\int_0^T g^2{}_2(t)\, dt}$$

Substituting the value of g_2 *(t), and simplifying,*

$$\varphi_2(t) = \frac{s_2(t) - s_{21}\varphi_1(t)}{\sqrt{\int_0^T [(s_2(t))^2 + (s_{21}\varphi_1(t))^2 - 2.s_2(t)s_{21}\varphi_1(t)]\, dt}} = \frac{s_2(t) - s_{21}\varphi_1(t)}{\sqrt{E_2 + s_{21}{}^2 - 2s_{21}{}^2}}$$

$$= \frac{s_2(t) - s_{21}\varphi_1(t)}{\sqrt{E_2 - s_{21}{}^2}}$$ Eq. 2

E_2 is the energy of the signal $s_2(t) = \int^T (s_2(t))^2\, dt, \int^T \varphi_1(t))^2\, dt = 1, \int^T s_2(t)\varphi_1(t)dt = s_{21}$

$$\int_0^T g_2(t)\, \varphi_1(t)\, dt = \int_0^T s_2(t)\, \varphi_1(t)\, dt - s_{21}\int_0^T [\varphi_1(t)]^2\, dt = s_{21} - s_{21} = 0$$

Hence, $g_2(t)$ is orthogonal to $\varphi_1(t)$ over the interval $[0,T]$.

$$\int_0^T \varphi_2(t)\,\varphi_2(t)\,dt = \int_0^T \frac{[\,s_2^2(t) - 2s_{21}\varphi_1(t)s_{21} + s_{21}\,(\varphi_1(t))^2\,]\,dt}{E_2 - s_{21}{}^2} = \frac{E_2 - s_{21}{}^2}{E_2 - s_{21}{}^2} = 1$$

$$\int_0^T \varphi_2(t)\,\varphi_1(t)\,dt = \int_0^T \frac{[\,s_2(t) - s_{21}\varphi_1(t)]\,\varphi_1(t)\,dt}{\sqrt{E_2 - s_{21}{}^2}} = \frac{s_{21} - s_{21}}{\sqrt{E_2 - s_{21}{}^2}} = 0$$

Hence $\varphi_2(t)$ and $\varphi_1(t)$ form an orthonormal set.

Phi 1 and Phi 2 are orthogonal to each other

$$s_1(t) = s_{11}\,\varphi_1(t) + 0.\,\varphi_2(t)$$

$$s_2(t) = s_{21}\,\varphi_1(t) + \sqrt{E_2 - s_{21}{}^2}\;\varphi_2(t)$$

Depending upon the above discussion, we may write the general form as given below.

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}(t)\varphi_j(t) \qquad s_{ij} = \int_0^T s_i(t)\,\varphi_j(t)\,dt,\; j = 1,2,3,\ldots\ldots i-1$$

$$\varphi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g^2{}_i(t)\, dt}}$$

General expression for basis function

Which form an orthonormal set.

The dimensions *N and M depend on one of the two possibilities.*

If $s_1(t)$, $s_2(t)$, …$s_M(t)$ *are linearly independent, N = M and*

If $s_1(t)$, $s_2(t)$, …$s_M(t)$ *are not linearly independent,* $N < M$ *and* $g_i(t) = 0$ *for* $i > N$.

Unlike Fourier series expansions of periodic signal, in Gram–Schmidt orthogonalization, we have not restricted the sinusoidal functions or sinc functions of time for the set of basis functions. The expansion of $s_i(t)$ *is not the approximations of the terms but the exact expression where the N terms and only N terms are significant.*

Summery Points for GOP

Let us summarize the steps to determine the orthonormal basis functions following the Gram-Schmidt Orthogonalization procedure: If the signal set {sj(t)} is known for j = 1, 2,….., M, 0 ≤ t

If the signal set {sj(t)} is known for j = 1, 2,….., M, 0 ≤ t<T,
Derive a subset of linearly independent energy signals, {si(t)},i = 1, 2,….., N ≤ M.

- Find the energy of s1(t) as this energy helps in determining the first basis function ϕ1(t), which is a normalized form of the first signal. Note that the choice of this 'first' signal is arbitrary.
- Find the scalar 's_{21}', energy of the second signal (E_2), a special function '$g_2(t)$' which is orthogonal to the first basis function and then finally the second orthonormal basis function $\phi_2(t)$

- Follow the same procedure as that of finding the second basis function to obtain the other basis functions.

Justification for G-S-O procedure

Methos 2: We show that any given set of energy signals, $\{s_i(t)\}$, $1 \le i \le M$ over $0 \le t < T$, can be completely described by a subset of energy signals whose elements are linearly independent.

To start with, let us assume that all $s_i(t)$ -s are not linearly independent. Then, there must exist a set of coefficients $\{a_i\}$, $1 < i \le M$, not all of which are zero, such that,

$$a_1 s_1(t) + a_2 s_2(t) + \ldots\ldots + a_M s_M(t) = 0, \quad 0 \le t < T$$

Verify that even if two coefficients are not zero, e.g. $a_1 \neq 0$ and $a_3 \neq 0$, then $s_1(t)$ and $s_3(t)$ are dependent signals.

Let us arbitrarily set, $a_M \neq 0$. Then,

$$s_M(t) = -\frac{1}{a_M}\left[a_1 s_1(t) + a_2 s_2(t) + \ldots\ldots + a_{M-1} s_{M-1}(t)\right] \qquad = -\frac{1}{a_M}\sum_{i=1}^{M-1} a_i s_i(t)$$

Next, we consider a reduced set with (M-1) signals {si(t)}, i = 1,2,….., (M – 1). This set may be either linearly independent or not. If not, there exists a set of {bi}, i = 1,2…, (M – 1), not all equal to zero such that,

$$\sum_{i=1}^{M-1} b_i s_i(t) = 0, \quad 0 \leq t < T$$

Again, arbitrarily assuming that $b_{M-1} \neq 0$, we may express $s_{M-1}(t)$

$$s_{M-1}(t) = -\frac{1}{b_{M-1}} \sum_{i=1}^{M-2} b_i s_i(t)$$

Now, following the above procedure for testing linear independence of the remaining signals, eventually we will end up with a subset of linearly independent signals. Let $\{s_i(t)\}$, i = 1, 2, …., $N \leq M$ denote this subset.

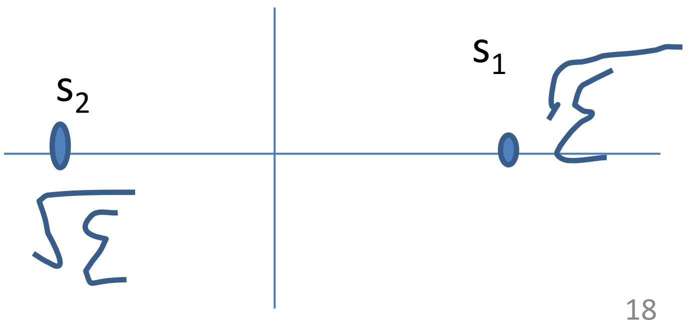
Concept of signal space

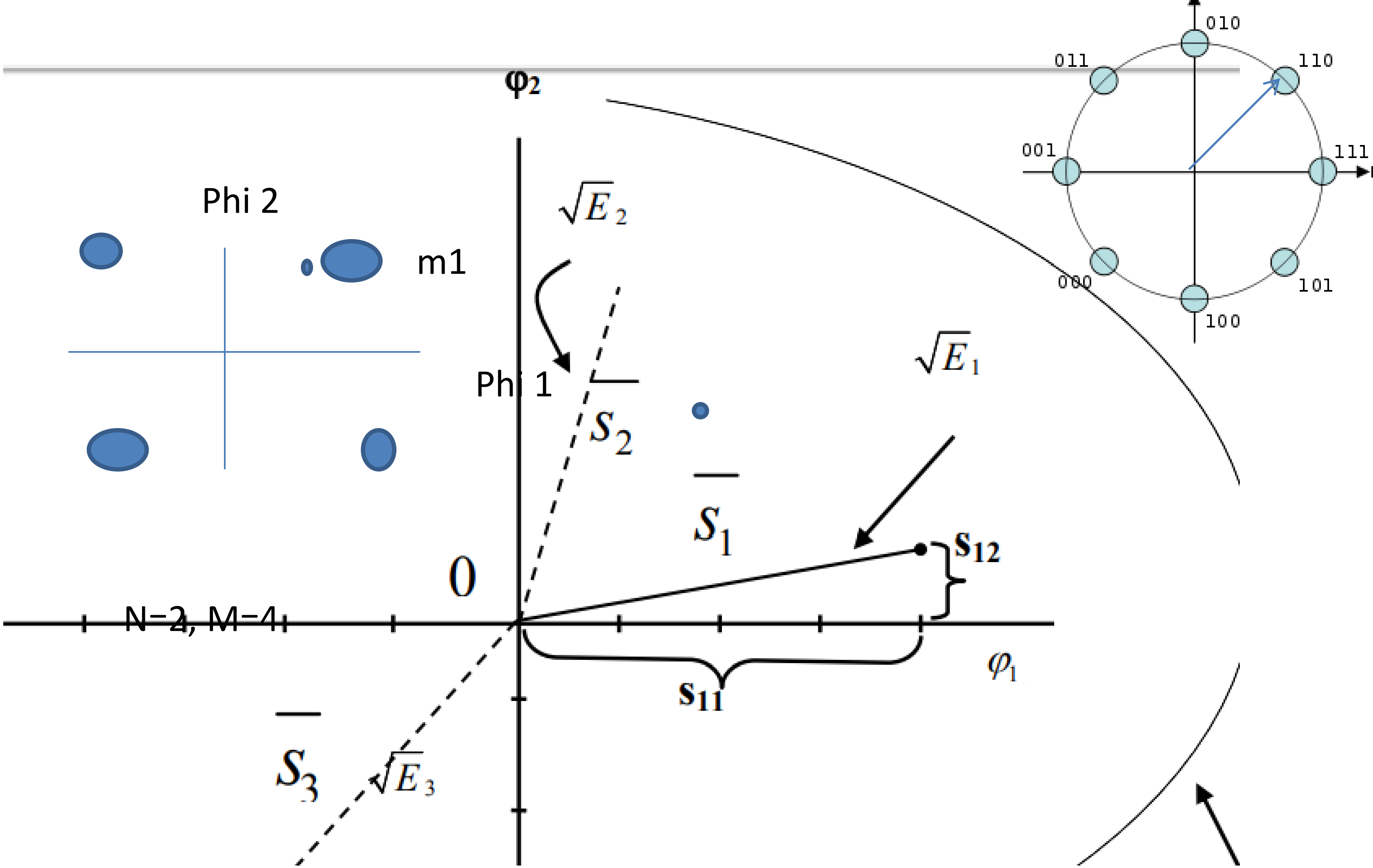
Now, we can represent a signal $s_i(t)$ as a column vector whose elements are the scalar coefficients s_{ij}, j = 1, 2, ….., N :

Now, we can represent a signal $s_i(t)$ as a column vector whose elements are the scalar coefficients s_{ij}, j = 1, 2, ….., N :

These M energy signals or vectors can be viewed as a set of M points in an N – dimensional Euclidean space, known as the 'Signal Space' Signal Constellation is the collection of M signals points (or messages) on the signal space.

$$\bar{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}_{1\times N} \quad ; \text{i}= 1, 2,……, \text{M}$$





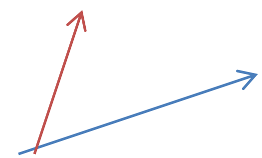
A **constellation diagram** is a representation of a signal modulated by a digital modulation scheme such as quadrature amplitude modulation or phase-shift keying.[1] It displays the signal as a two-dimensional *xy*-plane scatter diagram in the complex plane at symbol sampling instants.

Now, the length or *norm* of a vector is denoted as $\left\|\overline{s_i}\right\|$. The squared norm is the inner product of the vector:

$$\left\|\overline{s_i}\right\|^2 = \left(\overline{s_i}, \overline{s_i}\right) = \sum_{j=1}^{N} s^2_{ij}$$

The cosine of the angle between two vectors is defined as:

$$\cos\left(\text{angle between } \overrightarrow{s_\mathrm{i}} \text{ \& } \overrightarrow{s_\mathrm{j}}\right) = \frac{\left(\overrightarrow{s_\mathrm{i}}, \overrightarrow{s_\mathrm{j}}\right)}{\left\|\overrightarrow{s_\mathrm{i}}\right\| \left\|\overrightarrow{s_\mathrm{j}}\right\|}$$



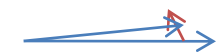
$\therefore \overrightarrow{s_\mathrm{i}}$ & $\overrightarrow{s_\mathrm{j}}$ are orthogonal to each other if $\left(\overrightarrow{s_\mathrm{i}}, \overrightarrow{s_\mathrm{j}}\right) = 0$.

If $\mathrm{E_i}$ is the energy of the i-th signal vector,

$$E_i = \int_0^T s_i^2(t)\,dt = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\varphi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\varphi_k(t)\right] dt$$

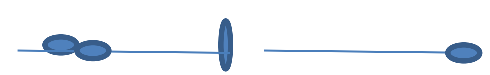
$$= \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}s_{ik}\int_0^T \varphi_j(t)\varphi_k(t)\,dt \quad \text{as } \{\varphi_j(t)\} \text{ forms an ortho-normal set}$$

$$= \sum_{j=1}^{N} s_{ij}^2 = \left\|\vec{s_i}\right\|^2$$

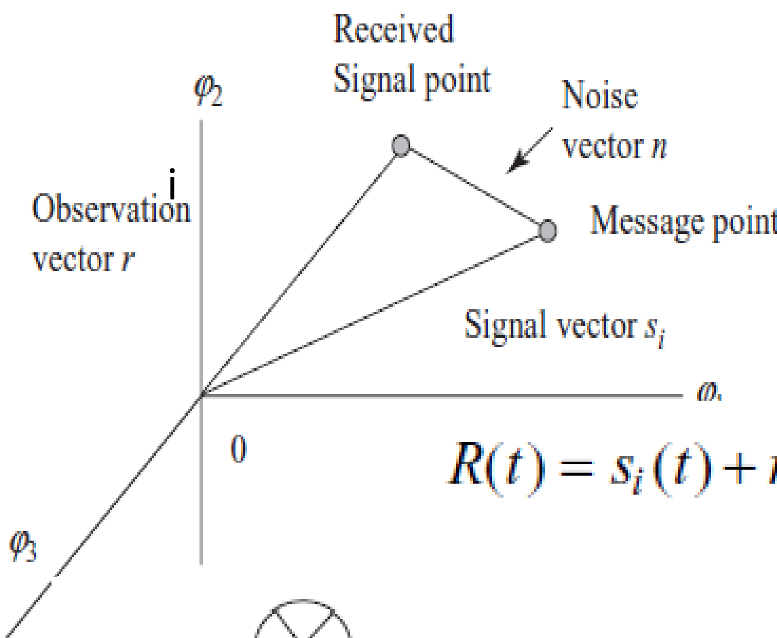


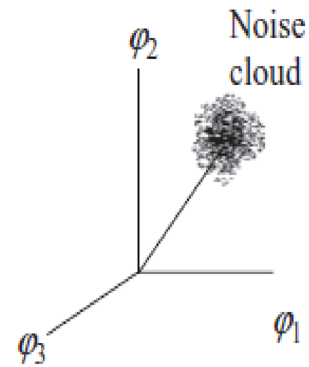
For a pair of signals $s_i(t)$ and $s_k(t)$, $\left\|\vec{s_i} - \vec{s_k}\right\|^2 = \sum_{j=1}^{N}\left(s_{ij} - s_{kj}\right)^2 = \int_0^T \left[s_i(t) - s_k(t)\right]^2 dt$

It may now be guessed intuitively that we should choose $s_i(t)$ and $s_k(t)$ such that the Euclidean distance between them, i.e. $\left\|\vec{s_i} - \vec{s_k}\right\|$ is as much as possible to ensure that their detection is more robust even in presence of noise. For example, if $s_1(t)$ and $s_2(t)$ have same energy E, (i.e. they are equidistance from the origin), then an obvious choice for maximum distance of separation is, $s_1(t) = -s_2(t)$.

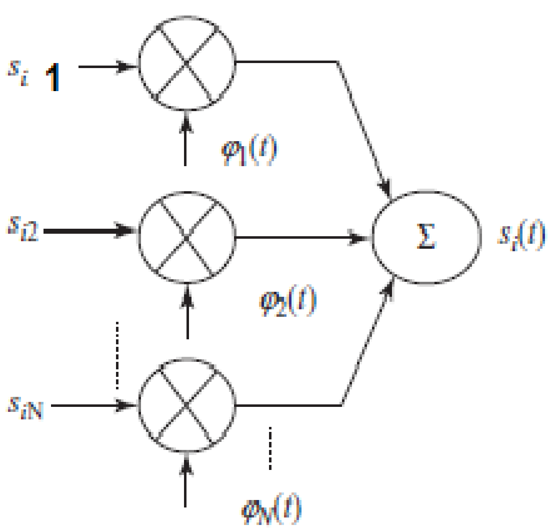


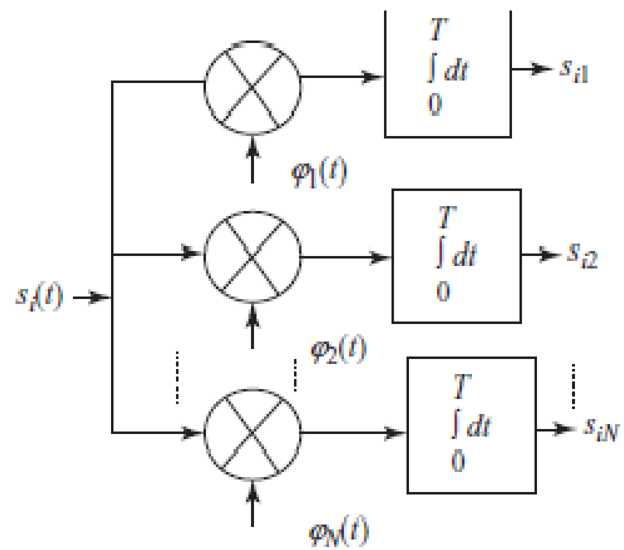
Signal Space for Signal Detection in a Receiver





$$R(t) = s_i(t) + n_i(t) \text{ for } 0 \leq t \leq T \text{ and } i = 1, 2, \ldots, M$$





Response of the Noisy Signal at the Receiver

- Any signal s_i (t) is represented in terms of orthonormal basis function φ_1 (t), φ_2 (t), …, φ_N (t), and its associated signal coefficients s_{i1}, s_{i2}, …, s_{iN}.
- At the receiver end if we derive the coefficients s_{i1}, s_{i2}, …, s_{iN} from the received signal, the transmitted signal would be obtained back.
-
- To get these coefficients, the received signal is multiplied by the basis functions φ_1 (t), φ_2 (t), …, φ_N (t), and then integrated, thus obtaining the contents of φ_1 (t), φ_2 (t), …, φ_N (t), in the received signal. This process is called correlation reception.

- During the transmission process, the signal gets corrupted with the Additive White Gaussian Noise (AWGN), so the received signal is basically the sum of the transmitted signal and the noise, which is a random process.

So, the output of the j^{th}, correlator is

$$R_j = \int_0^T R(t)\,\varphi_j(t)\,dt = s_{ij} + N_j, \quad j = 1, 2, \ldots N \text{ and} \qquad s_{ij} = \int_0^T s_i(t)\,\varphi_j(t)\,dt$$

N_j is the th sample value of the random channel noise n(t), AWGN

$$N_j = \int_0^T n(t)\,\varphi_j(t)\,dt$$

Consider a random process R'(t) whose sample value if r'(t) related to the received signal R(t) as ,

$$r'(t) = R(t) - \sum_{j=1}^{N} R_j\,\varphi_j(t) = s_i(t) + n(t) - \sum_{j=1}^{N} (s_{ij} + N_j)\,\varphi_j(t)$$

$$= n(t) - \sum_{j=1}^{N} N_j\,\varphi_j(t) = n'(t)$$

The sample function r'(t) solely depends on the channel noise.

Correlator Output: Statistical measure

Let W_j denote the random variable represented by the sample value w_j produced by the jth correlator in response to the white Gaussian noise component $w(t)$. The random variable W_j has zero mean, because the noise process $W(t)$ represented by $w(t)$ in the AWGN model Consequently, the mean of X_j depends only on s_{ij}, as shown by

$$\mu_{X_j} = E[X_j] = E[s_{ij} + W_j] = s_{ij} + E[W_j] = s_{ij}$$

To find the variance of X_j, we note that

$$\begin{aligned} \sigma_{X_j}^2 &= \text{var}[X_j] = E[(X_j - s_{ij})^2] \\ &= E[W_j^2] \end{aligned} \tag{5.35}$$

$$\begin{aligned} \sigma_{X_j}^2 &= E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_j(u)du\right] \\ &= E\left[\int_0^T \int_0^T \phi_j(t)\phi_j(u)W(t)W(u)dtdu\right] \end{aligned} \qquad W_j = \int_0^T W(t)\phi_j(t)dt$$

Interchanging the order of integration and expectation:

$$\begin{aligned} \sigma_{X_j}^2 &= \int_0^T \int_0^T \phi_j(t)\phi_j(u)E[W(t)W(u)]dtdu \\ &= \int_0^T \int_0^T \phi_j(t)\phi_j(u)R_W(t, u)dtdu \end{aligned}$$

$R_W(t, u)$ is the autocorrelation function of the noise process $W(t)$.

depends only on the time difference $t - u$.

$W(t)$ is white with a constant power spectral density $N_0/2$,

$$R_W(t, u) = \frac{N_0}{2}\delta(t-u)$$

$$\sigma^2_{X_j} = \frac{N_0}{2}\int_0^T\int_0^T \phi_j(t)\phi_j(u)\ \delta(t-u)dtdu$$

$$= \frac{N_0}{2}\int_0^T \phi_j^2(t)dt$$

$$\boxed{\sigma^2_{X_j} = \frac{N_0}{2} \quad \text{for all } j}$$

This important result shows that all the correlator outputs denoted by X_j with $j = 1, 2, \ldots, N$, have a variance equal to the power spectral density $N_0/2$ of the noise process $W(t)$.

since the $\phi_i(t)$ form an orthogonal set, we find that the X_j are mutually

$$\boxed{\text{cov}[X_jX_k] = E[(X_j - \mu_{X_j})(X_k - \mu_{X_k})] = E[(X_j - s_{ij})(X_k - s_{ik})] = E[W_jW_k]}$$

$$= E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_k(u)du\right]$$

$$= \int_0^T \int_0^T \phi_j(t)\phi_k(u)R_W(t, u)dtdu$$

$$= \frac{N_0}{2}\int_0^T \int_0^T \phi_j(t)\phi_k(u)\ \delta(t - u)dtdu$$

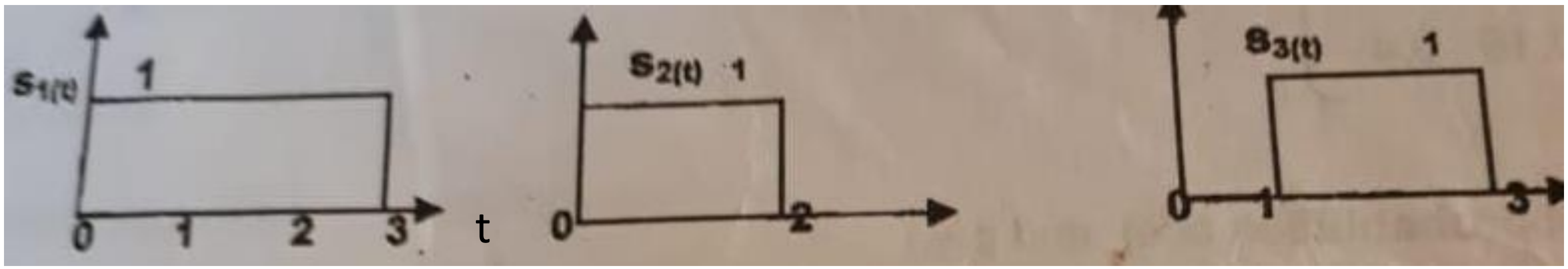
$$= \frac{N_0}{2}\int_0^T \phi_j(t)\phi_k(t)dt$$

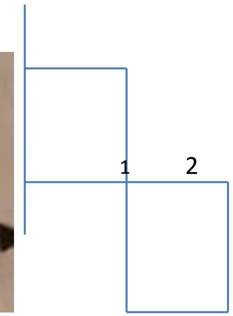
$$= 0, \qquad j \neq k$$

$N_{0/2}$ for $j=k$

Problem solving using GSOP

Use the GSOP to express the functions in the following figure in terms of orthonormal components φ_n (t)





GSO is used to convert non-orthogonal signals into orthonormal signals.
Those non-orthogonal signals can be expressed completely with the help of those orthonormal basis vectors.

The signal sets are defined as

S_1 (t) = 1, $0 \leq t \leq 3$
0, Other wise

S_2 (t) = 1, $0 \leq t \leq 2$
0, Other wise

S_3 (t) = 1, $1 \leq t \leq 3$
0, Other wise

So the maximum number of components that can be derived from these set of signals is 3. i=1,2,3

φ_n (t)

GSOP

Let e_n (t) be the orthonormal set for n= 1, 2 and 3

$e_1(t) = S_1(t) / \,||\, S_1(t) \}||$ (the norm of S_1 (t) $= \sqrt{E_1} =$ Sqrt of energy of the signal $E_1 = \int_0^T S_1(t)^2\, dt$

$E_1 = 3$, so $e_1(t) = S_1(t)/\sqrt{3}$ and thus S_{11} = component of $S_1(t)$ on $e_1(t) = \sqrt{3}$

$S_1(t) = S_{11}\, e_1(t)$ eq. (1)

$S_2(t) = S_2(t) - S_{21}\, e_1(t)$ (2) $S_{21} = \int_0^T S_2(t)\, e_1(t)\, dt = 2/\sqrt{3}$

$e_2(t) = S_2(t) / \,||\, S_2 \,||\, = S_2(t) / 2/\sqrt{3}$ (2)

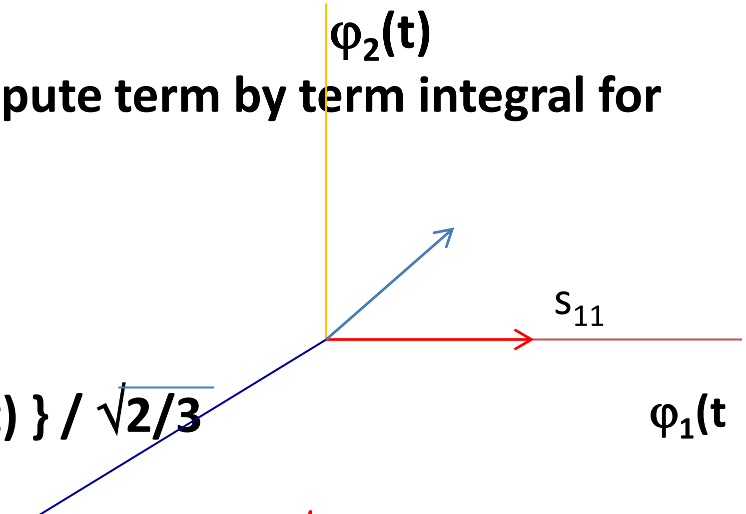
Similarly , you need to find $S_3(t) = S_3(t) - S_{31}\, e_1(t) - S_{32}\, e_2(t)$

Where $S_{31} = = \int_0^3 S_3(t)\, e_1(t)\, dt = 2/\sqrt{3}$, $S_{32} = \int_0^3 S_3(t)\, e_2(t)\, dt = -1/\sqrt{6}$

GSOP

Now, $e_3(t) = S_3(t) \,/\, ||S_3(t)||$ where we need to find $||S_3(t)||$

$= \sqrt{\int_0^3 S_3(t)^2 \, dt}$ **$= 1/\sqrt{2}$ (you need to compute term by term integral for each $S_3(t)$ components**



So . We obtain $e_1(t) = \varphi_1(t) =$ $S_1(t) \,/\, \sqrt{3}$

$e_2(t) = \varphi_2(t) = \{S_2(t) - 2/\sqrt{3}\, \varphi_1(t)\} \,/\, \sqrt{2/3}$

and **$e_3(t) = \varphi_3(t) = \{ S_3(t) - 2/\sqrt{3}\, \varphi_1(t) + 1/\sqrt{6}\, \varphi_2(t) \} \,/\, 1/\sqrt{2}$**

So finally we can express, $S_1(t) = \sqrt{3}\, \varphi_1(t) + 0.\, \varphi_2(t) + 0 .\, \varphi_3(t)$

$S_2(t) = 2/\sqrt{3}\, \varphi_1(t) + \sqrt{2/3}\, \varphi_2(t) + 0.\, \varphi_3(t)$

and $S_3(t) = 2/\sqrt{3}\, \varphi_1(t) - 1/\sqrt{6}\, \varphi_2(t) + 1/\sqrt{2} .\, \varphi_3(t)$

Maximum Likelihood Estimation

Maximum Likelihood Estimation seeks the solution that "best" explains the observed dataset

$$\theta^{ML} = \underset{\theta}{\operatorname{argmax}}\ P(X|\theta)$$

Or

$$= \underset{\theta}{\operatorname{argmax}}\ \log P(X|\theta)$$

Translation: "select as our maximum likelihood parameters those parameters that resulted in a maximization of the probability of the observation given those parameters".

i.e. **we seek to maximize** $P(X|\theta)$ **over all possible** θ

This is sometimes called the *maximum likelihood criterion*

Log likelihood is often very handy as we often would otherwise need to deal with a long product of terms…

$$\theta^{\mathrm{ML}} = \operatorname{argmax} \log \prod_{i=1}^{k} \mathrm{P}(x_i|\theta)$$

$$= \operatorname{argmax} \sum_{i=1}^{k} \log \mathrm{P}(x_i|\theta)$$

This often comes about because there are multiple outcomes that need to be considered

ML Decision Rule and Boundary

- The conditional probability density functions $f_r(r|m_i)$, i = 1,2,3, …M, are basically the characterization of an AWGN channel.
- Its derivation leads to a functional dependence on the observation vector r, given the message symbol m_i is transmitted.
- At the receiver, given the observation vector r, and the requirement is the estimation of the message symbol m_i

The received signal point is wandered around the message point in a completely random fashion as it may reside anywhere inside the Gaussian distributed noise cloud, centered on the message point

We introduce likelihood function being defined as

$$L(m_i) = f_r(r \mid m_i), \;\; \mathrm{i} = 1,2,3, \ldots, M$$

in logarithmic form as $l(m_i) = \log L(m_i)$,

as the probability density function is always non-negative and the logarithmic function is always monotonically increasing function.

Now we are ready to a state of signal detection problem:

Given the observation vector r,

a mapping is to be performed from r to estimate $\hat{m}$ from the transmitted symbol m_i

in a way that would minimize the error probability during the decision making process.

Given the observation vector r, we make the decision $\hat{m} = m_i$, the probability of error in the decision process is defined by $P_e(m_i|r)$.

$$P_e(m_i \mid r) = P(m_i \text{ not sent}|r)$$
$$= 1 - P(m_i \text{ sent}|r)$$

the optimum decision rule is made.

Set $\hat{m} = m_i$, if $P(m_i \text{ sent}|r) \geq P(m_k \text{ sent}|r)$ for all $k \neq i$

where $k = 1, 2, \ldots, M$.

This is called Maximum a Posteriori Probability (MAP) rule.

In Bayesian statistics, the posterior probability of a random event or an uncertain proposition is the conditional probability that is assigned after the relevant evidence is taken into account.

MAP rule may also be expressed more explicitly in terms of *the a priori probabilities of the transmitted signals and in terms of likelihood functions.*

Using Bay's criterion and ignoring possible ties in the decision making process, the MAP rule can be restated as follows:

Set $\hat{m} = m_i$, if

$$\frac{p_k f_r(r \mid m_k)}{f_r(r)} \text{ is maximum for } k = i$$

Where p_k is the a priori probability of transmitting symbol m_k,

$f_r(r|m_k)$ is the conditional probability density function of the random observation vector R given the transmitted message symbol m_k,

When all the source symbols are transmitted are equally likely, then $p_k = p_i$

The conditional probability $f_r(r|m_k)$ *has the one-to-one mapping to the log-likelihood function* $l(m_k)$**. Accordingly the decision rule called the maximum likelihood rule can be stated as,**

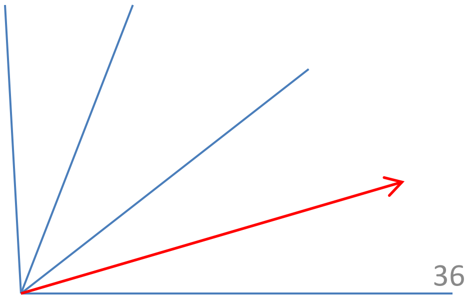
$$\text{Set } \hat{m} = m_i \text{ if}$$

$$l(m_k) \text{ is maximum for } k = i$$

It is useful to have a graphical interpretation of the maximum likelihood decision rule. If *Z denotes the N-dimensional observation space for all observation vectors r, the total observation space is partitioned into M-decision regions if* $m = m_i$, $i = 1,2, \ldots M$. *The decision regions are* $Z_1, Z_2, Z_3, \ldots Z_M$. *The decision rule is restated as*

Observation vector r lies in region Z_i *if*

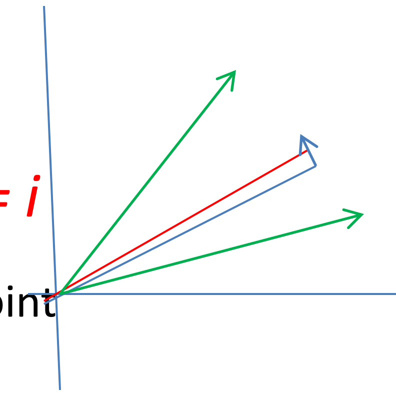
$l(m_k)$ is maximum for k = i



if observation vector r falls on the boundary of two decision regions, solution comes from fare coin tossing

The Euclidian distance between the observation point *r and message point* s_k *is represented by* $||r - s_k||$. *Accordingly the maximum likelihood decision rule is stated as,*

Observation vector *r lies in region* Z_i *if*
the Euclidean distance $||r - s_k||$ *is minimum for* $k = i$



which implies that the message point is the closest to the received signal point

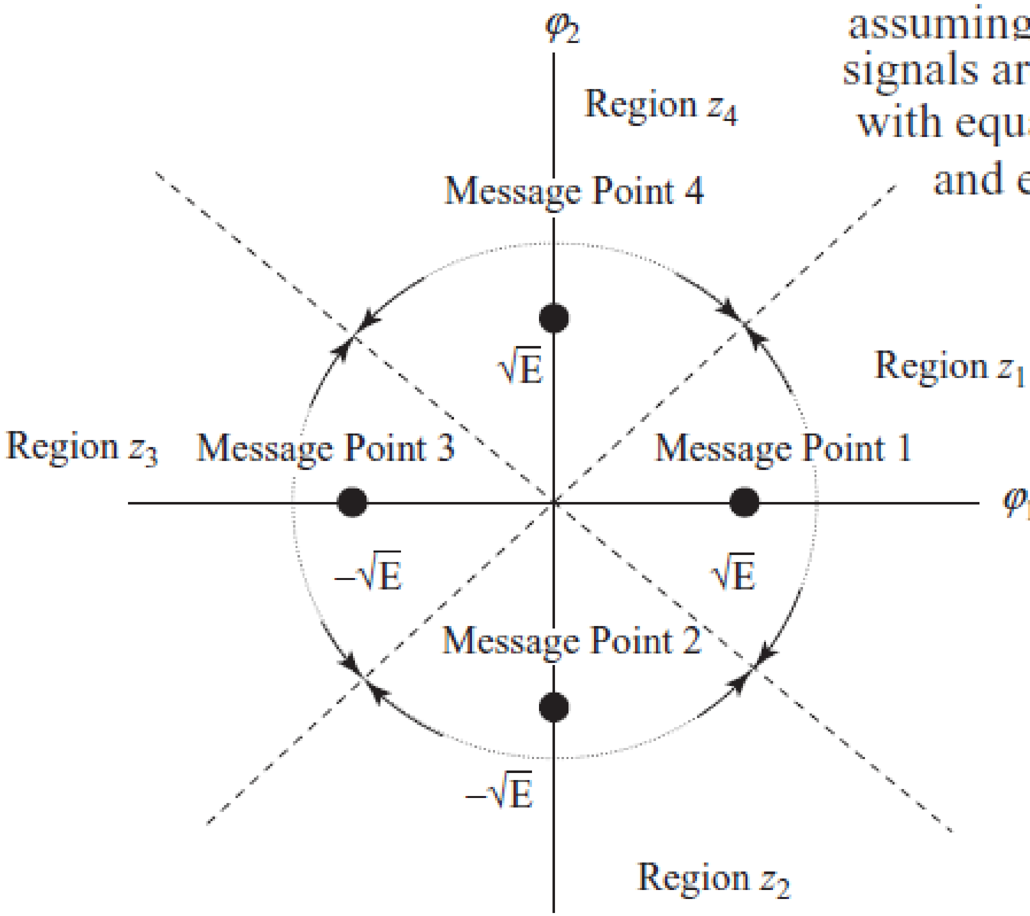
$$\sum_{j=1}^{N} (r_j - s_{kj})^2 = \sum_{j=1}^{N} r_j^2 - 2\sum_{j=1}^{N} r_j \, s_{kj} + \sum_{j=1}^{N} s_{kj}{}^2$$

a modified decision rule can be drawn as,

Observation vector r lies in region Z_i, if

$$\sum_{j=1}^{N} r_j s_{kj} - E_k/2 \text{ is maximum for } k = i$$

assuming the signals are transmitted with equal energy, E, and equal probability.



decision regions for $M = 4$ signals and $N = 2$ dimensions, We introduce likelihood function

$$L(m_i) = f_r(r \mid m_i), \;\; \mathrm{i} = 1, 2, 3, \ldots, M$$

in logarithmic form as $l(m_i) = \log L(m_i)$,

Pbobability of error in estimating transmitted signal m_i from the received signal r

$$P_e(m_i \mid r) = P(m_i \text{ not sent} \mid r)$$
$$= 1 - P(m_i \text{ sent} \mid r)$$

This rule requires the receiver to determine the probability of transmission of a message from the received vector.

The optimum decision rule is made as

$$\text{Set } \hat{m} = m_i \text{ , if } P(m_i \text{ sent} \mid r) \geq P(m_k \text{ sent} \mid r) \text{ for all } k \neq i$$

This is called Maximum a Posteriori Probability (MAP) rule.

If a priori probabilities for transmitted symbols are known , then MAP rule will be restated by Baysian rule as

$$\underbrace{\Pr(m_i \mid \bar{r})}_{\substack{\text{a posteriori prob.} \\ \text{of 'm}_i\text{' given } \bar{r}}} \; \underbrace{\Pr(\bar{r})}_{\substack{\text{joint probability} \\ \text{of } \bar{r}}} = \underbrace{\Pr(\bar{r} \mid m_i)}_{\substack{\text{a priori prob.} \\ \text{of } \bar{r} \text{ given 'm}_i\text{'}}} \underbrace{\Pr(m_i)}_{\frac{1}{M}}$$

$$\text{Set } \hat{m} = m_i \text{ , if}$$

$$\frac{p_k f_r(r \mid m_k)}{f_r(r)} \text{ is maximum for } k = i$$

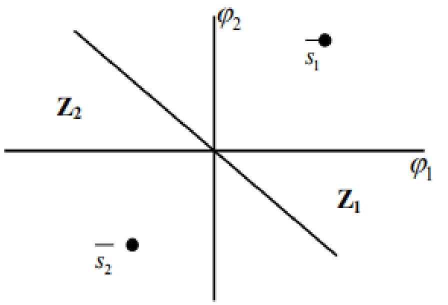
When all the source symbols are transmitted are equally likely, then $p_k = p_i$

The conditional probability $f_r(r|m_k)$ has the one-to-one mapping to the log-likelihood function ***$l(m_k)$. Accordingly the decision rule called the maximum likelihood rule can be stated as,***

$$\text{Set } \hat{m} = m_i \text{ if}$$

$$l(m_k) \text{ is maximum for } k = i$$

The decision regions are $Z_1, Z_2, Z_3, \ldots Z_M$. The decision rule is restated as



Observation vector r lies in region Z_i *if*

$l(m_k)$ is maximum for $k = i$

Observation vector *r lies in region Z_i if*
the Euclidean distance $||r - s_k||$ *is minimum for $k = i$*

$$\sum_{j=1}^{N} (r_j - s_{kj})^2 = \sum_{j=1}^{N} r_j^2 - 2\sum_{j=1}^{N} r_j\, s_{kj} + \sum_{j=1}^{N} s_{kj}^{\,2}$$

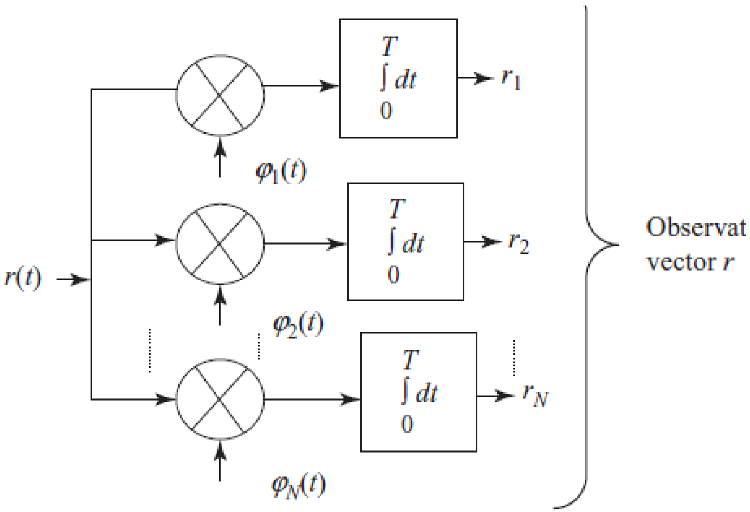
a modified decision rule can be drawn as,

Observation vector r lies in region Z_i, if

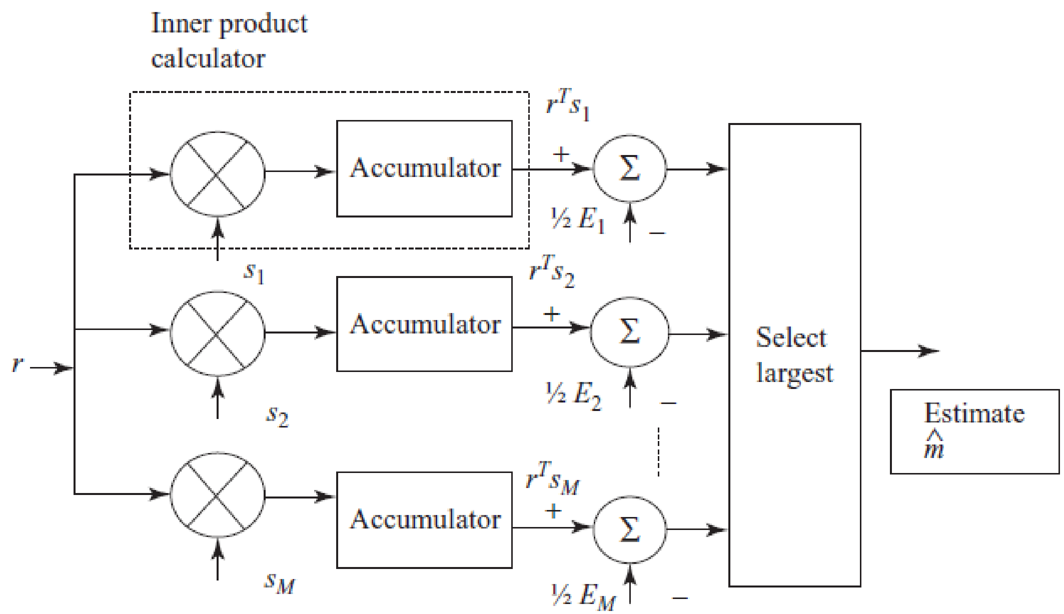
$$\sum_{j=1}^{N} r_j s_{kj} - E_k/2 \text{ is maximum for } k = i$$

Optimum Correlation Receiver with AWGN Channel

Transmitted signals s_1 *(t)*, s_2 *(t), …*s_M *(t). The Optimum receiver has two parts:*



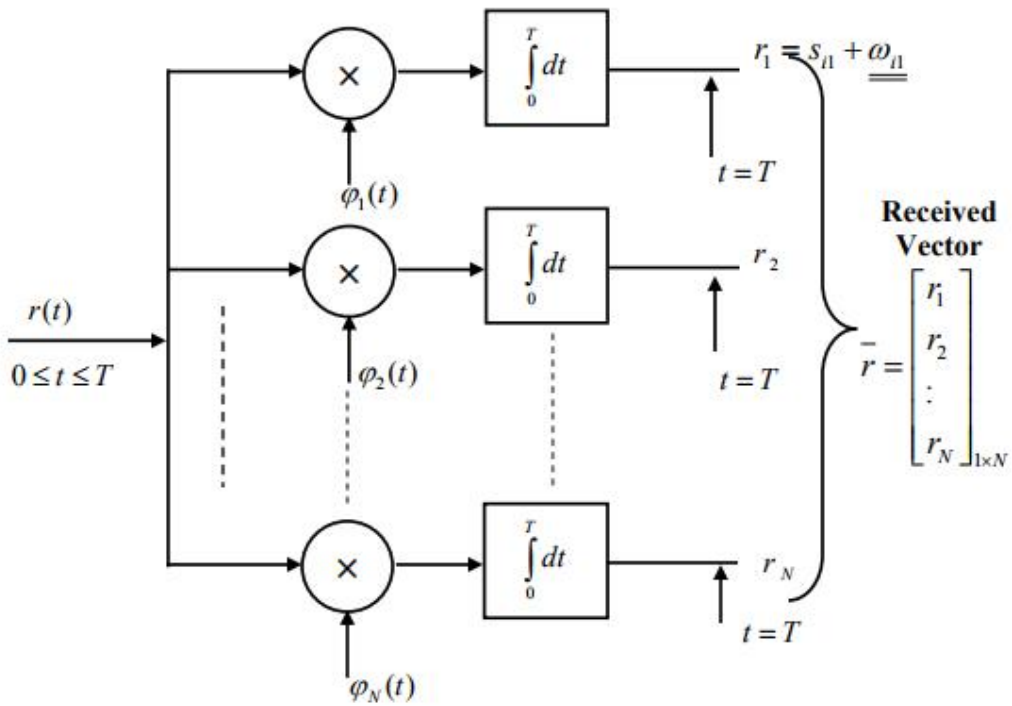
M correlators or product integrators to produce observation vector r



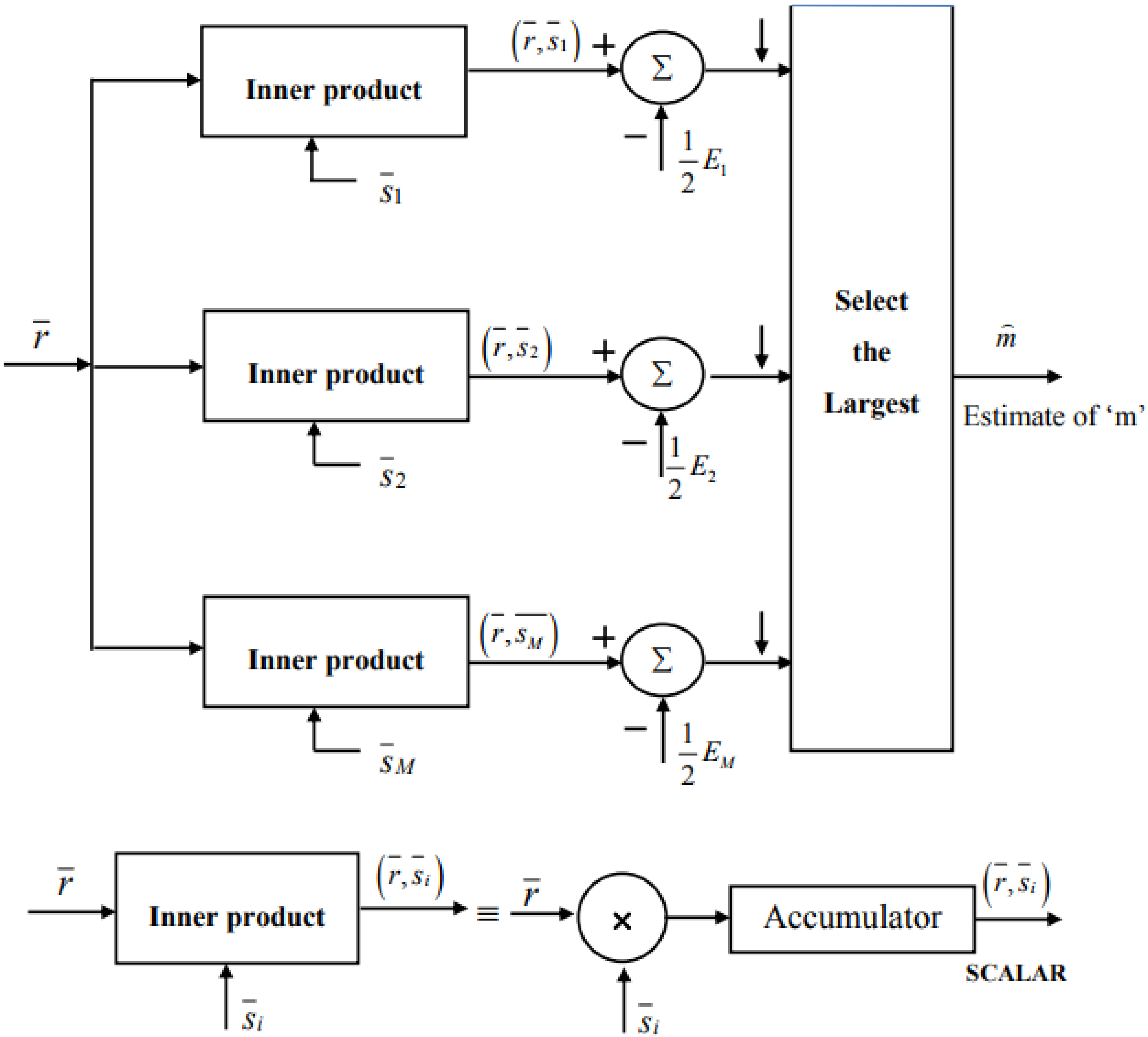
Transmission decoder based on MLE, it provides the best estimation of the transmitted signal from observation point

Correlation Receiver

A Correlation Receiver, consisting of a Correlation Detector and a Vector Receiver implements the M L decision rule by, (a) first finding r with a correlation detector and then (b) computing the metric and taking decision in a vector receiver.



The structure of a Correlation Detector for determining the received vector $\bar{r}$ from the received signal r(t)



Block schematic diagram for the Vector Receiver

Feature of the received vector

The j-th element of r , which is obtained at the output of the j-th correlator once in T second, can be expressed as:

$$r_j = \int_0^T r(t)\Phi_j(t)dt = \int_0^T [s_i(t)+w(t)]\Phi_j(t)dt$$

$$= s_{ij} + w_j \ ; \quad j=1,2,\ldots, N$$

Here w_j is a Gaussian distributed random variable with zero mean

the mean of the correlator out put is,

$E[r_j] = E[s_{ij}+w_j] = E[s_{ij}] = s_{ij} = m_{rj}$, say. We note that the mean of the correlator out put is independent of the noise process. However, the variances of the correlator outputs are dependent on the strength of accompanying noise:

$$Var[r_j] = \sigma_{r_j}^2 \quad = E\left[(r_j - s_{ij})^2\right] = E[w_j^2] \quad = E\left[\int_0^T w(t)\Phi_j(t)dt \int_0^T w(u)\Phi_j(u)du\right]$$

$$= E\left[\int_0^T\int_0^T \Phi_j(t)\Phi_j(u).w(t)w(u)dtdu\right]$$

$$\sigma_{rj}^2 = \int_0^T\int_0^T \Phi_j(t)\Phi_j(u)E[w(t).w(u)]dtdu$$

$$= \int_0^T\int_0^T \Phi_j(t)\Phi_j(u)R_w(t,u)dtdu$$

Interchanging integration and expectation operators

Here, R_w (t-u) is the auto correlation of the noise process. As we have learnt earlier, additive white Gaussian noise process is a WSS random process and hence the auto-correlation function may be expressed as, $R_w(t,u) = R_w(t-u)$ and further,

$R_w(t-u) = \frac{N_0}{2}\delta(t-u)$, where '$N_o$' is the single-sided noise power spectral density in Watt/Hz. So, the variance of the correlator output now reduces to:

$$\sigma_{rj}^2 = \frac{N_0}{2}\int_0^T\int_0^T \Phi_j(t)\Phi_j(u)\delta(t-u)dtdu$$

$$= \frac{N_0}{2}\int_0^T \Phi_j^2(t)dt = \frac{N_0}{2}$$

It is interesting to note that the variance of the random signals at the out puts o N correlators are a) same, b) independent of information-bearing signal waveform an dependent only on the noise psd.

Now the likelihood function for $s_i(t)$, and the ML decision rule can be expressed in terms of the output of the correlation detector.

The likelihood function for 'm_i' = $Pr(\bar{r}|m_i) = f_{\bar{r}}(\bar{r}|m_i) = f_{\bar{r}}(\bar{r}|s_i(t))$, where, $f_r(r|m_i)$ is the conditional pdf of 'r' given 'm_i'.

$$f_{\bar{r}}\left(\bar{r}\middle|m_i\right)=\prod_{j=1}^{N} f_{r_j}\left(r_j\middle|m_i\right),\quad \mathrm{i}=1,2,\ldots,\mathrm{M}$$

where, $f_{r_j}\left(r_j\middle|m_i\right)$ is the pdf of a Gaussian random variable with mean s_{ij} & var. = $\sigma_{r_j}^2$ =

$$\frac{N_0}{2},\ \text{i.e.,}\ f_{r_j}\left(r_j\middle|m_i\right)=\frac{1}{\sqrt{2\pi\sigma_{r_j}^2}}.e^{-\frac{(r_j-s_{ij})^2}{2\sigma_{ij}^2}}$$

$$f_{\bar{r}}\left(\bar{r}\middle|m_i\right)=\left(\pi N_0\right)^{-\frac{N}{2}}.\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(r_j-s_{ij}\right)^2\right],\quad \mathrm{i}=1,2,\ldots\ldots,\mathrm{M}$$

This generic expression is of fundamental importance in analyzing error performance of digital modulation schemes

Probability of error

Considering a digital binary transmission system where symbols 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude *A and equal duration Tb (bit duration). The channel noise is AWGN with zero mean and power spectral density N0 /2. Within the signaling duration 0 ≤ t ≤ Tb, the received signal is considered of the form,*

$$r(t) = \begin{cases} +A + n(t) & \text{when symbol 1 is sent} \\ -A + n(t) & \text{when symbol 0 is sent} \end{cases}$$

Given the noisy signal $r(t)$, the receiver is required to make a decision in each signaling interval whether transmitted symbol is a 1 or a 0. Consider Fig. 5.7 as the receiving system for the binary transmitted signal.

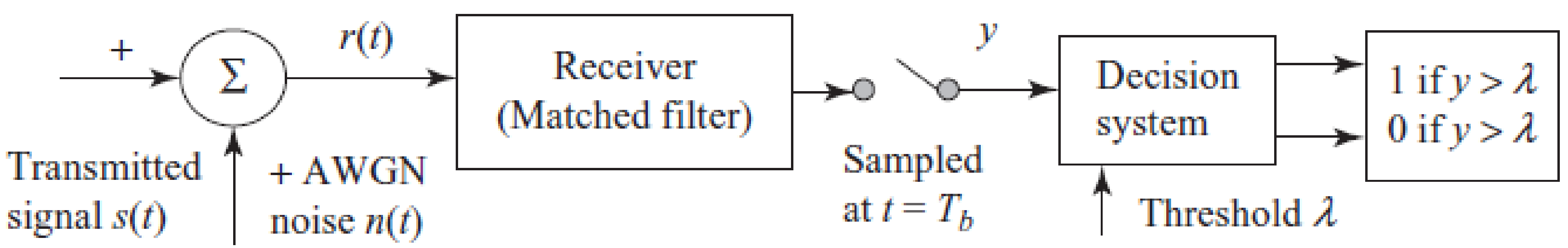
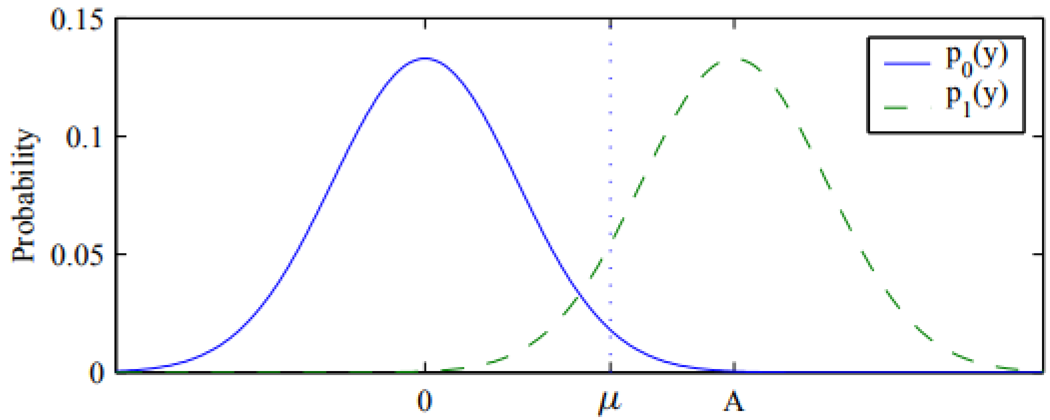


Fig. 5.7 *Receiver system for binary transmitted signal*

a filter which is matched to a known signal ϕ(t), 0 ≤t ≤ T is characterized by an impulse response h(t) which is a time reversed and delayed version of ϕ( )t i.e. h(t) = − ϕ (t-T)



The quantity μ is a threshold which we would usually choose somewhere between 0 and A.

when a signal is present, the density of y is

$$p_1(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-A)^2/(2\sigma^2)}.$$

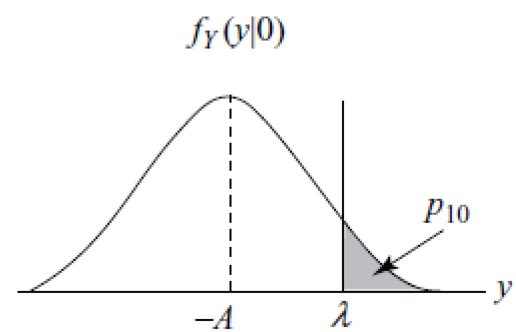
$$p_0(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/(2\sigma^2)}.$$

In the decision making process, there is always involvement of two types of error:
Symbol 1 is chosen when 0 was actually sent, we call this error as the Type I error.1.
Symbol 0 is chosen when 1 was actually sent; we call this error as the Type II error.

Let p_{10} *denote the conditional probability of error, given that symbol 0 was sent and* p_{01} *is the conditional probability of error, given that symbol 1 was sent. These probabilities are defined as follows:*

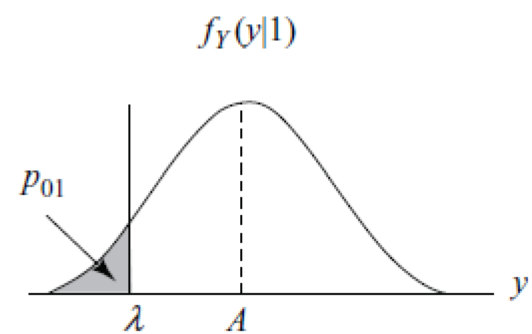
$$p_{10} = P(y > \lambda \mid \text{symbol 0 was sent})$$

$$= \int_{\lambda}^{\infty} f_Y(y \mid 0)\, dy \qquad = 1/\sqrt{\pi N_0 / T_b} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0 / T_b}\right) dy$$



$$p_{01} = P(y > \lambda \mid \text{symbol 1 was sent})$$

$$= \int_{-\infty}^{\lambda} f_Y(y \mid 1)\, dy \qquad = 1/\sqrt{\pi N_0 / T_b} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y-A)^2}{N_0 / T_b}\right) dy$$



$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

The upper bound of the complementary error function: $\mathrm{erfc}(u) < \exp(-u^2) / \sqrt{\pi}\, u$

In terms of error function, p_{10} can be expressed as, where $z = (y + A) / \sqrt{N_0 / T_b}$

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{(A+\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz = 1/2\ \mathrm{erfc}\,(A + \lambda) / \sqrt{N_0 / T_b}$$

Similarly, error function p_{01} can be expressed as, where $z = (A - y) / \sqrt{N_0 / T_b}$

$$p_{01} = \frac{1}{\sqrt{\pi}} \int_{(A-\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz = 1/2\ \mathrm{erfc}\,(A - \lambda) / \sqrt{N_0 / T_b}$$

$$\begin{aligned} p_e &= p_0\ p_{10} + p_1\ p_{01} \\ &= p_0 /2\ \mathrm{erfc}\,(A + \lambda) / \sqrt{N_0 / T_b}\,) + p_1 /2\ \mathrm{erfc}\,(A - \lambda) / \sqrt{N_0 / T_b}\,) \end{aligned}$$

a priori probabilities of transmitting symbols 0 and 1 are p_0 and p_1 respectively.

For this special case of $p_{01} = p_{10,}$ *the channel is binary symmetric and the average probability of symbol error pe becomes,*

$$p_e = 1/2 \text{ erfc } (A / \sqrt{N_0 / T_b})$$

The transmitted symbol energy per bit is $E_b = A^2 T_b$

$$p_e = 1/2 \text{ erfc } (\sqrt{E_b / N_0})$$

Error Probability

Let that observation space Z is partitioned according to maximum likelihood decision rule into set of M regions $\{Z_i\}$, $i = 1,2, \ldots M$. When symbol m_i or signal vector s_i is transmitted, let the observation vector r is received. Error will occur when the observation vector r does not lie within region Z_i associated with the message point represented by s_i. Averaging of all possible transmission, the average probability of error p_e is defined as,

$$p_e = \sum_{i=1}^{M} p_i \; P(r \text{ does not lie in } Z_i \mid m_i \text{ is sent})$$

$$= 1/M \sum_{i=1}^{M} P(r \text{ does not lie in } Z_i \mid m_i \text{ is sent})$$

$$= 1 - 1/M \sum_{i=1}^{M} P(r \text{ lies in } Z_i \mid m_i \text{ is sent})$$

In terms of likelihood function,

$$p_e = 1 - 1/\, M \sum_{i=1}^{M} \int_{Zi} f_r(r \mid m_i)\, dr$$

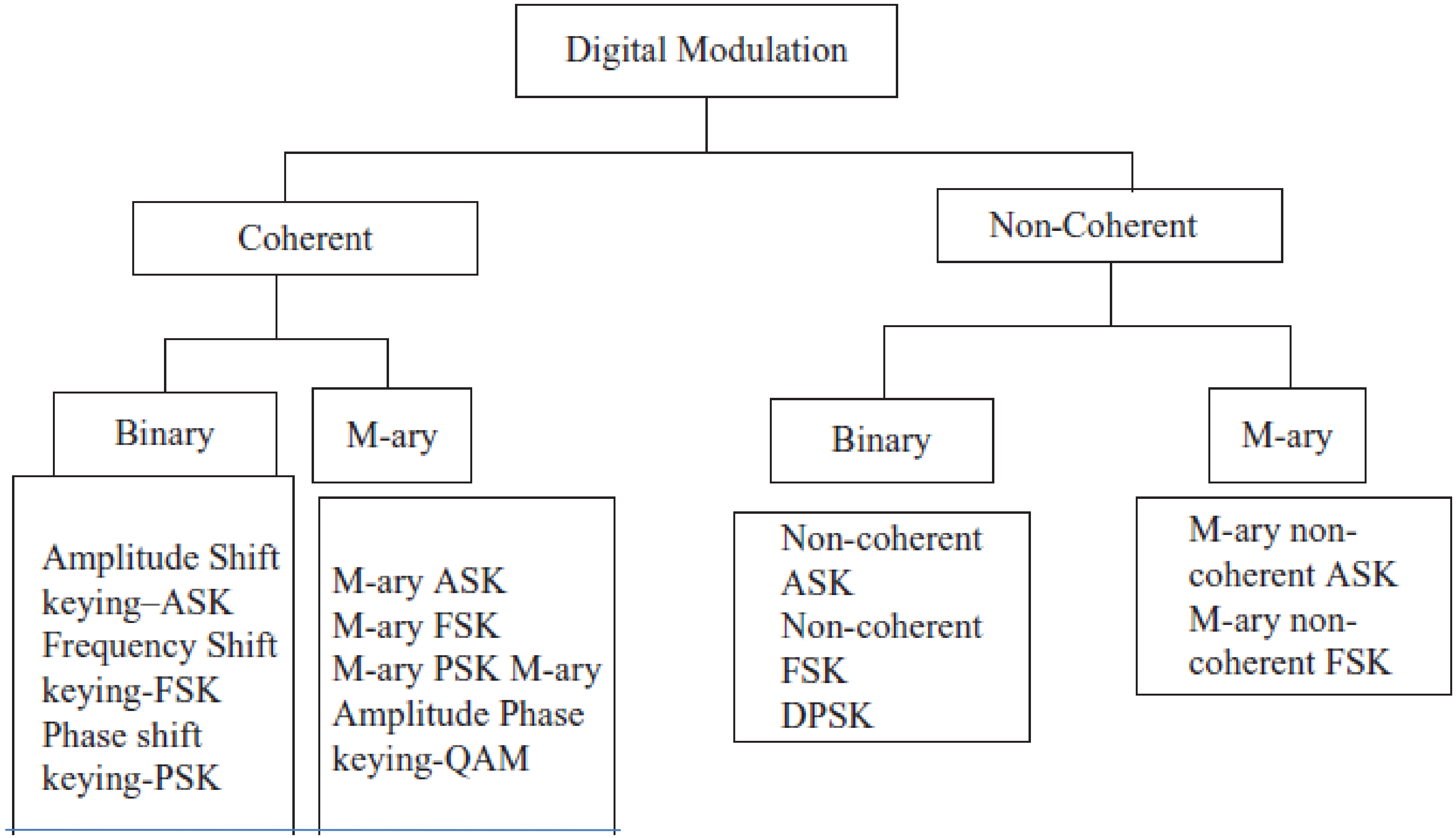
Digital Modulation

The mapping of data bits into signal waveforms for transmission is the domain of digital modulation.

At the receiver, the demodulator recovers the data bits from the received signal waveforms.

The simplest modulation is the binary one where +1 bit value is mapped into one specific waveform while the –1 bit value maps into other specific waveform.

The most important technique in digital communication is the M-ary modulation where group of k bits can be expressed into a symbol which again is mapped into one out of a set of $M = 2^k$ *waveforms.*



In an M-ary signaling scheme, we may send any one of *M possible signals* $s_1(t), s_2(t), \ldots s_M(t)$, *during each signaling interval of duration T. The number of possible signals* $M = 2^n$, *n is an integer. The symbol duration* $T = nT_b$, *where* T_b *is the bit duration.*

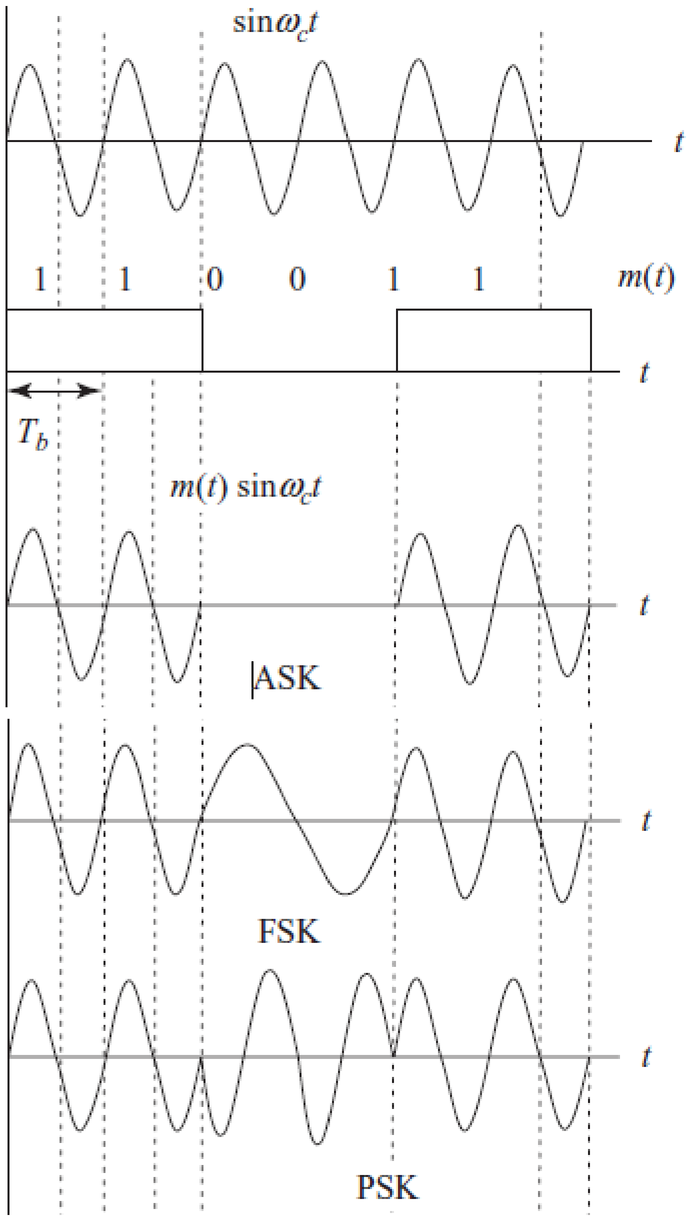
M-ary communication

In baseband data transmission, these signals are generated in different form by changing amplitude, phase and frequency of a sinusoidal carrier in *M discrete steps.*

In this way, M-ary ASK, M-ary PSK and M-ary FSK digital modulation schemes are obtained.

A special form of modulation namely M-ary quadrature-amplitude modulation (QAM) is obtained by a hybrid technique.
M-ary signaling is used when there is a requirement of conserving bandwidth at the cost of higher power.

When the bandwidth of the channel is less than the required value, M-ary modulation is used for maximum bandwidth efficiency.

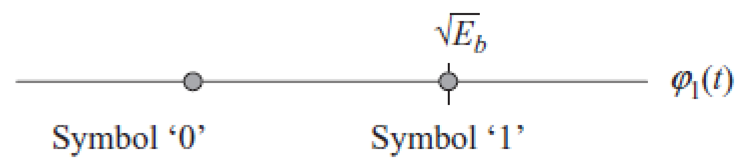


$$s(t) = \sqrt{2P_s}\ \sin\omega_c t$$

the symbol 1

$P_s = E_b/T_b$, unit bit signal energy, T_b is the bit duration.

$$s(t) = \sqrt{P_s T_b}\ \varphi_1\ (t) = \sqrt{E_b}\varphi_1(t)$$



So, the distance between the two message points

$$d = \sqrt{E_b}\ .$$

Coherent BPSK

$$s_1(t) = \sqrt{2E_b/T_b}\,\cos(2\pi f_c\, t)$$ symbols 1

$$s_2(t) = -\sqrt{2E_b/T_b}\,\cos(2\pi f_c\, t)$$ and 0

for $0 \le t \le T_b$

Each transmitted bit contains number of cycles of the carrier wave.

The carrier frequency $f_c = n \times 1/T_b$, where n is an integer,

for BPSK only one basis function with unit energy is defined:

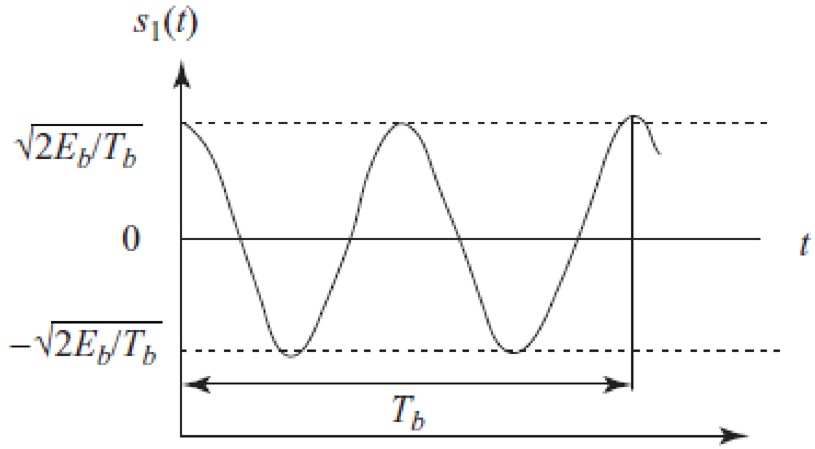
$\varphi_1(t) = \sqrt{2/T_b}\,\cos(2\pi f_c t)$ for $0 \le t \le T_b$ and thus $s_1(t)$ and $s_2(t)$ can be defined as,

$$s_1(t) = \sqrt{E_b}\,\varphi_1(t) \text{ and } s_2(t) = -\sqrt{E_b}\,\varphi_1(t) \quad \text{for} \quad 0 \le t \le T_b$$

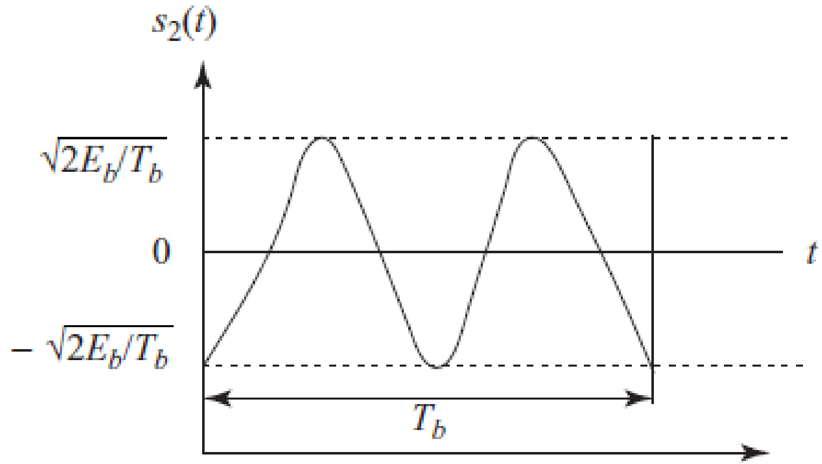
BPSK system has one dimensional signal space and two message points.

$$s_{11} = \int_0^{T_b} s_1(t)\,\varphi_1(t)\,dt = +\sqrt{E_b}$$

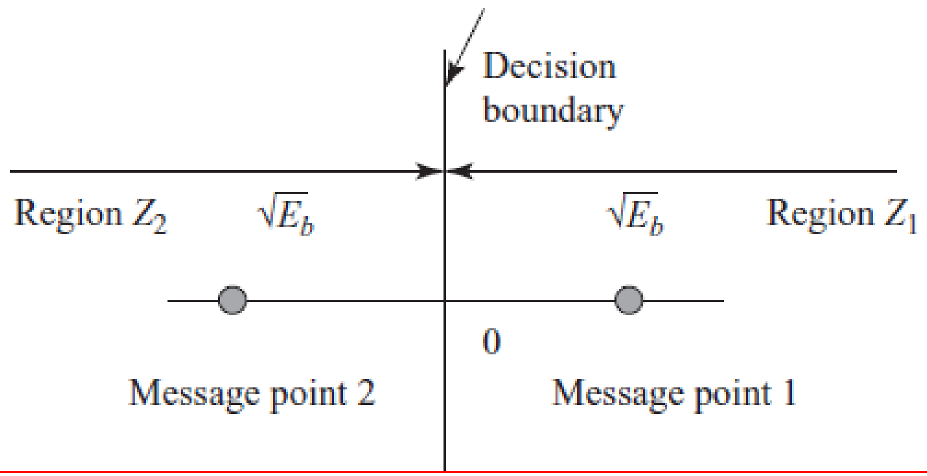
$$s_{12} = \int_0^{T_b} s_2(t)\,\varphi_1(t)\,dt = -\sqrt{E_b}$$



(a)



(b)



Thus for BPSK two message points lie at $s_{11} = \sqrt{E_b}$ and $s_{12} = -\sqrt{E_b}$ corresponding to the signals $s_1(t)$ (for 1 bit) and $s_2(t)$ (for 0 bit) respectively.

Error Probability of BPSK

As discussed previously, let define two decision regions Z1 and Z2 for BPSK signals corresponding to symbol 1 and symbol 0 respectively.

To make this decision two kinds of erroneous decision may occur as discussed in earlier section. Let the decision region associated with symbol 1 or signal $s_1(t)$ *is described as,*

Z1 : 0< r1< ∞ where the observed signal element r_1 *is related to the received signal r (t).*

$$r_1 = \int_0^{T_b} r(t)\,\varphi_1(t)\,dt$$

The conditional probability density function of random variable *R1, given that symbol 0 (signal s2 (t)) is transmitted is defined as,*

$$\boxed{\begin{aligned} f_{R1}(r_1 \mid 0) &= \frac{1}{\sqrt{\pi N_0}} \exp\left(-1/N_0 (r_1 - s_{21})^2\right) \\ &= \frac{1}{\sqrt{\pi N_0}} \exp\left(-1/N_0 (r_1 + \sqrt{E_b})^2\right) \end{aligned}}$$

$$f_{\bar{r}}\left(\bar{r}\middle|m_i\right)=(\pi N_0)^{-\frac{N}{2}}.\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(r_j-s_{ij}\right)^2\right],\quad i=1,2,\ldots\ldots,M$$

The conditional probability

$$p_{10}=\int_0^\infty f_{R_1}(r_1\mid 0)\,dr_1$$ in favor of the symbol 1, given that symbol 0 was transmitted

$$=\frac{1}{\sqrt{\pi N_0}}\int_0^\infty \exp\left(-1/N_0\,(r_1+\sqrt{E_b})^2\right)dr_1 \qquad (5.60)$$

$$p_{10}=\frac{1}{\pi}\int_{\sqrt{E_b/N_0}}^{\infty}\exp(-z^2)\,dz \qquad Let\ z=(1/\sqrt{N_0})(r_1+\sqrt{E_b})$$

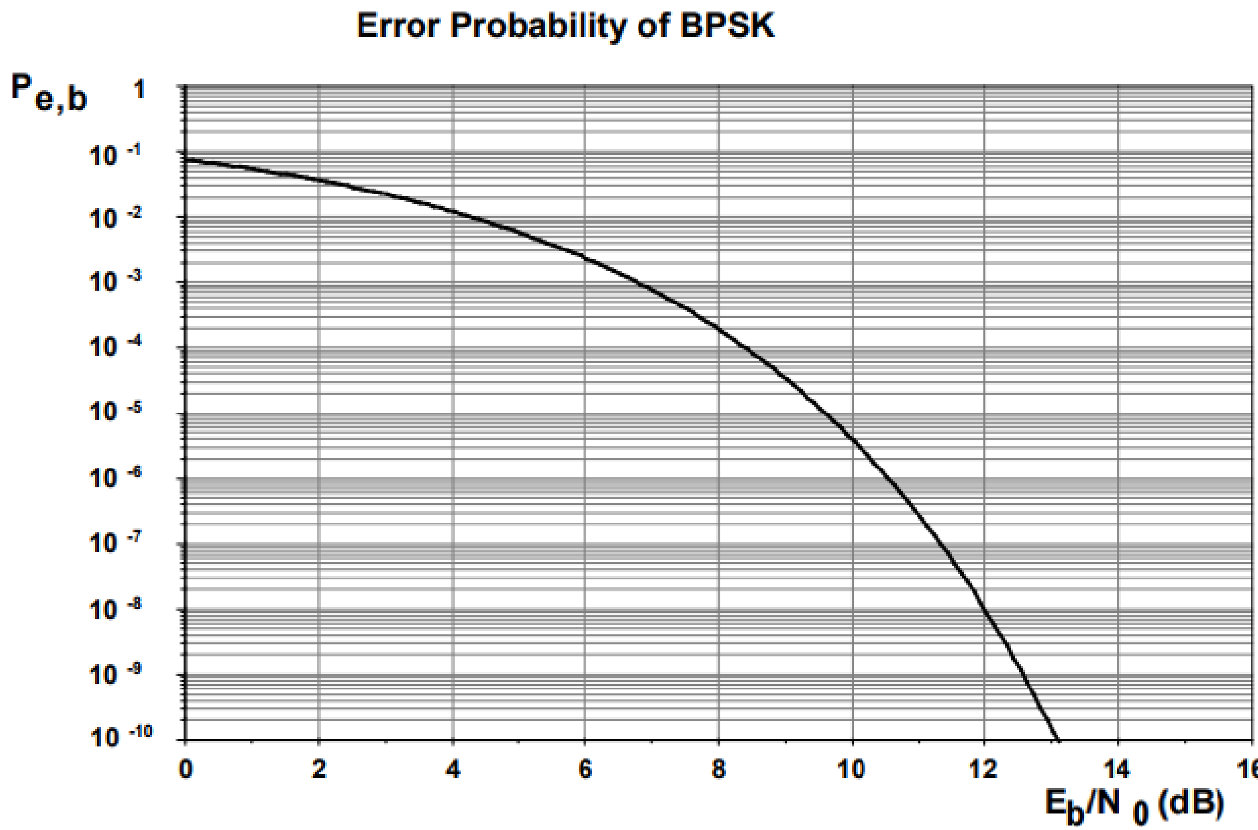
$$=1/2\ \mathrm{erfc}\left[\sqrt{E_b/N_0}\right]$$

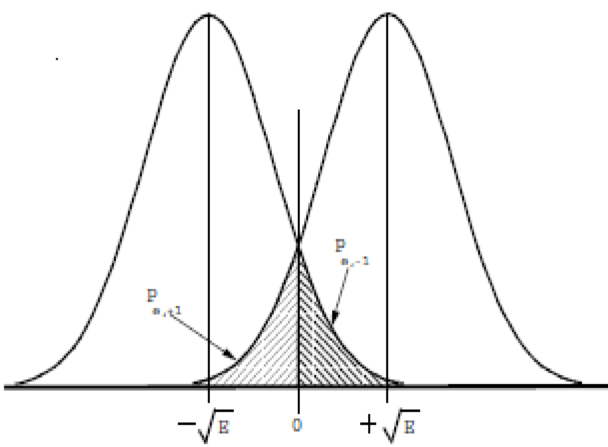
conditional probability $p_{01}=1/2\ \mathrm{erfc}\left[\sqrt{E_b/N_0}\right]$ in favor of the symbol 0, given that symbol 1 was transmitted

$$p_e=p_0\ p_{10}+p_1\ p_{01}$$

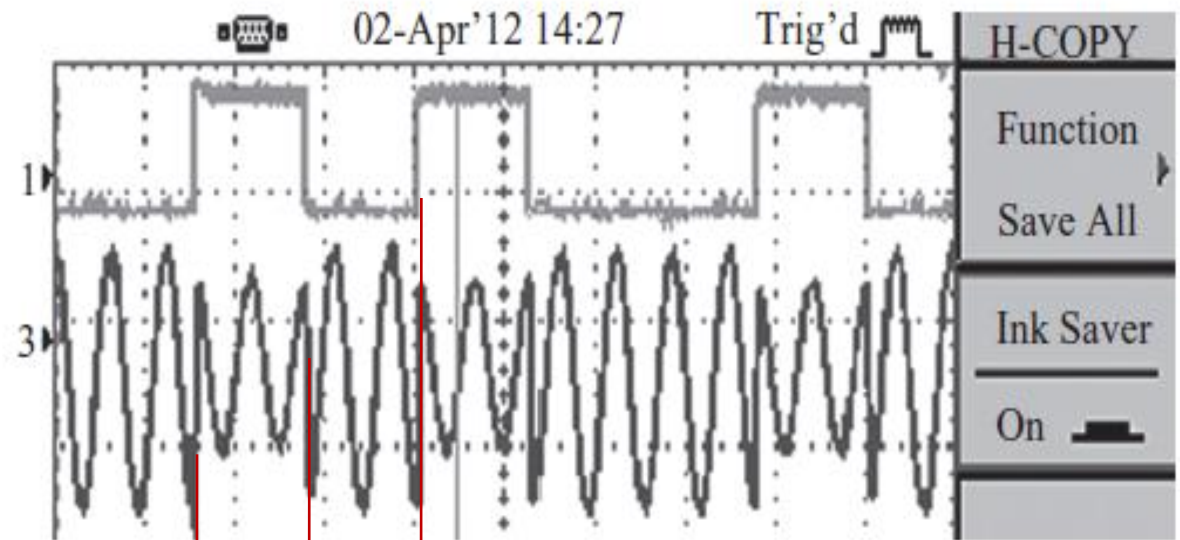
$$p_e=1/2\ \mathrm{erfc}\left[\sqrt{E_b/N_0}\right]$$

Average prob. Of error with p_0 and p_1 =1/2 , equally likely



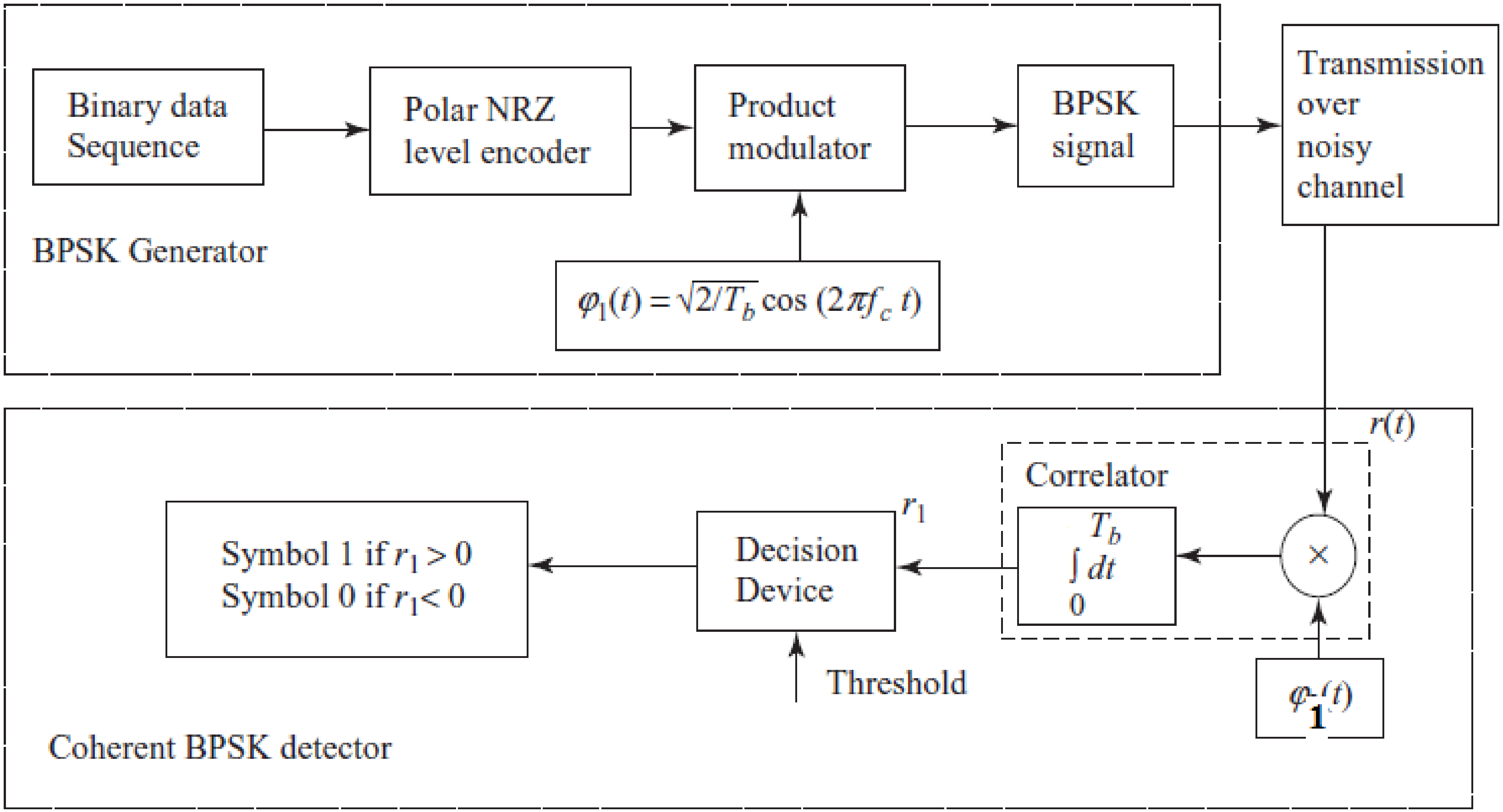


As we increase the energy per bit E_b for a specified noise spectral density, the message points $\sqrt{E_b}$ and $-\sqrt{E_b}$ corresponding to 1 and 0 move further apart. So, probability of error detection will be reduced. At this stage it may be inferred that erroneous detection can be avoided by increasing the power of the transmitter, however, it is not always the only remedy from real engineering point of view.



experimentally observed output of the BPSK modulator with a data stram of 01010010.

BPSK Generation and Detection



Power spectrum BPSK

In the BPSK waveform, there are rectangular pulses of amplitude $\pm\sqrt{2E_b/T_b}$ over the duration $0 \le t \le T_b$ depending on whether we have symbol 1 or 0. So, the symbol shaping function $g(t)$ of the BPSK signal can be defined as,

$$\boxed{g(t) = \begin{cases} \sqrt{2\,E_b\,/T_b} & 0 \le t \le T_b \\ 0 & \text{Otherwise} \end{cases}}$$

The input binary waves are random in nature. If we consider that 1 and 0 transmissions are equally likely and the symbols transmitted at different times are statistically independent, then over large number of symbols transmission, the power spectral density can be averaged over the duration of the transmission.

The energy spectral density of the base signal is found by getting the Fourier transform of the g (*t) which has a sinc function distribution.*

Example – Rectangular Pulse

- Derive the Fourier transform of the rectangular pulse function shown.

$$x(t)=\begin{cases} A & \text{for } -\tau/2<t<\tau/2 \\ 0 & \text{elsewhere} \end{cases}$$

$\omega = 2\pi f$

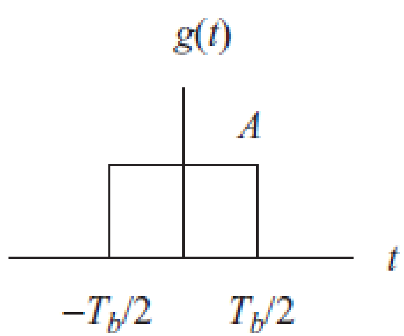
$$X(f)=2\int_0^{\frac{\tau}{2}} A\cos\omega t\,dt=\frac{2A}{\omega}\sin\omega t\Big|_0^{\frac{\tau}{2}}=\frac{2A}{\omega}\sin\frac{\omega\tau}{2}$$

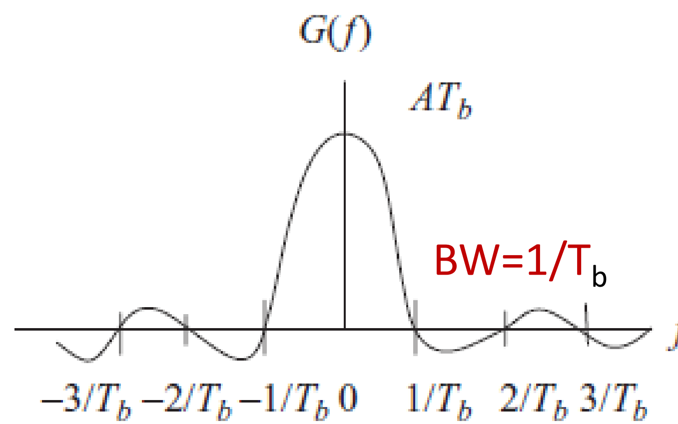
$$X(f)=A\tau\frac{\sin\pi f\tau}{\pi f\tau}$$

$G(f) = AT_b \text{ Sinc } (\ \text{pi } fT_b)$

$= AT_b \text{ Sinc } (wT_b\ /2)$

$A = \text{Sqrt } (\ 2E_b/T_b)$

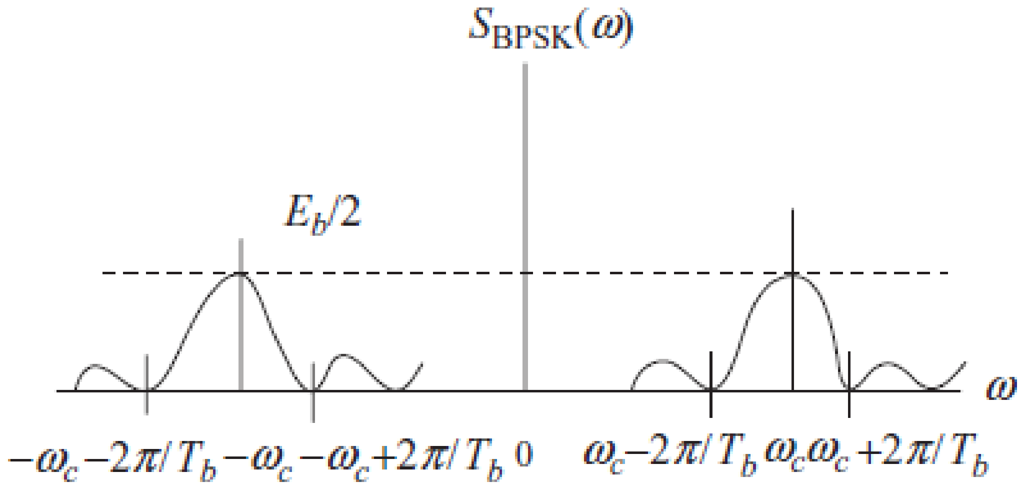




The energy spectral density (ESD) $= |G(\omega)|^2 = A^2 T_b^{\,2} \operatorname{sinc}^2(\omega T_b/2) = 2E_b T_b \operatorname{sinc}^2(\omega T_b/2)$

The power spectral density (PSD) $S(\omega) = \mathrm{ESD}/T_b = 2E_b \operatorname{sinc}^2(\omega T_b/2)$

The BPSK signal $\quad g(t)\cos(2\pi f_c t) \overset{FT}{\Leftrightarrow} 1/2\,[G(\omega+\omega_c)+G(\omega-\omega_c)]$



Bandwidth $2/T_b$

M-ary Modulation

In M-ary signaling, the source symbol can take *M possible number of integers and the amplitude or phase of the carrier takes on one of M possible values.*

The information carried by each M-ary symbol is $\log_2 M$ *binary digits.*

For M-ary ASK M possible Amplitude

MPSK – M possible Phase of 2pi/M radians

Thus $S \in \{\pm 1, \pm 2, \ldots\ldots \pm (M-1)\}$, there are M possible waveforms; during a symbol interval T_s one of the waveforms is selected for transmission. The M-ary communication may be multi-amplitude (MASK) where M symbols are transmitted by M-pulses $\pm g(t), \pm 3g(t), \pm 5g(t), \ldots\ldots \pm (M-1)g(t)$. In case of multi-phase (MPSK) signaling M pulses with phases of successive pulse of $2\pi/M$ radians would be transmitted.

5.16.1 M-ary phase shift keying (MPSK)

The phase of the carrier takes on one of M possible values such as $\theta_i = 2\pi(i-1)/M$, where $i = 1,2,3,\ldots,M$. For each signaling duration T_s, one of M possible waveform is represented as,

$$s_i(t) = \sqrt{2E_s/T_s}\cos\left[2\pi f_c t + 2\pi(i-1)/M\right] \text{ over } 0 \le t \le T_s \qquad (5.68)$$

$$i = 1, 2, \ldots M$$

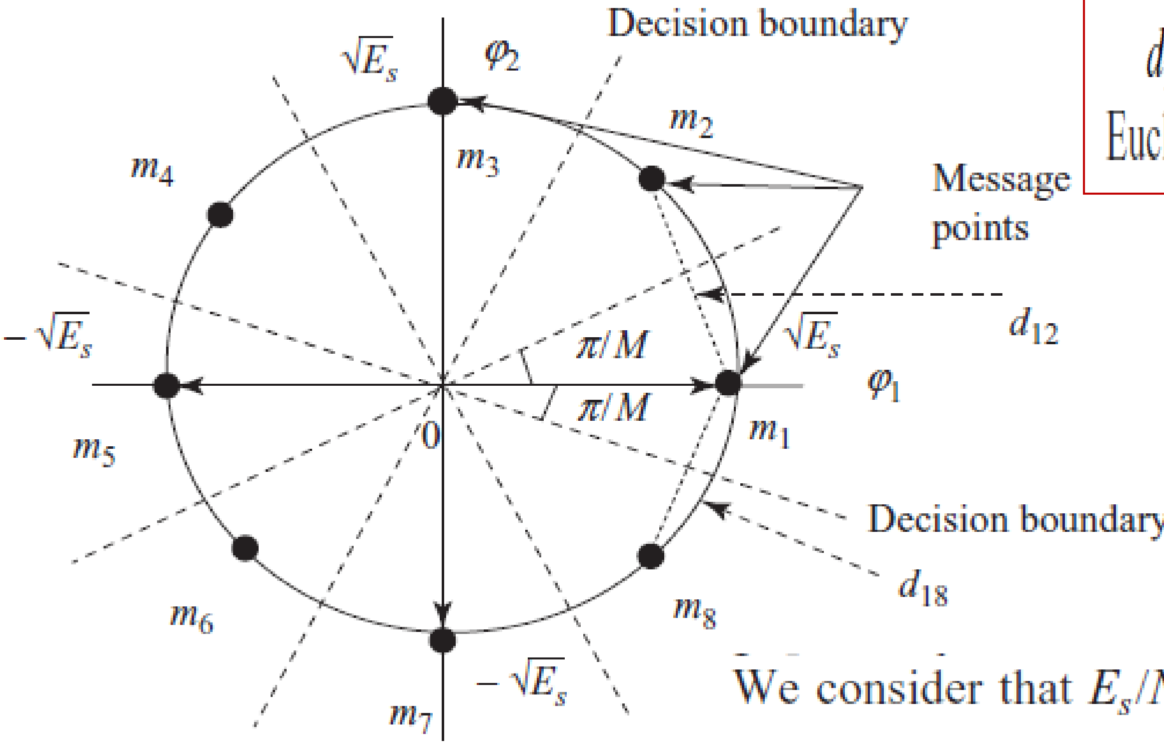
where, E_s is the energy of the symbol $= \int_{-\infty}^{+\infty} s_i^2(t)\,dt$

The carrier frequency $f_c = n/T_s$, for any integer n. Each of $s_i(t)$ may be expanded in terms of two basis functions $\varphi_1(t)$ and $\varphi_2(t)$.

$$\varphi_1(t) = \sqrt{2/T_s}\cos 2\pi f_c t \quad \text{and} \quad \varphi_2(t) = \sqrt{2/T_s}\sin 2\pi f_c t \quad \text{for } 0 \le t \le T_s$$

$i = 1,2,\ldots M$ Thus the signal constellation of MPSK is two-dimensional.

$E_s = nE_b$ and $T_s = nT_b$, Two dimensional signal space diagram
M message points lie on the circle of radius $\sqrt{E_s}$



$$d_{12} = d_{18} = 2\sqrt{E_s}\sin(\pi/M)$$

Euclidean distance between two message points

We consider that E_s/N_0 ratio is large enough to detect the message point m_1 on the decision boundary.

$$P_e = \text{erfc}\left[\sqrt{E_s/N_0}\sin(\pi/M)\right]$$ We assume $M >= 4$.

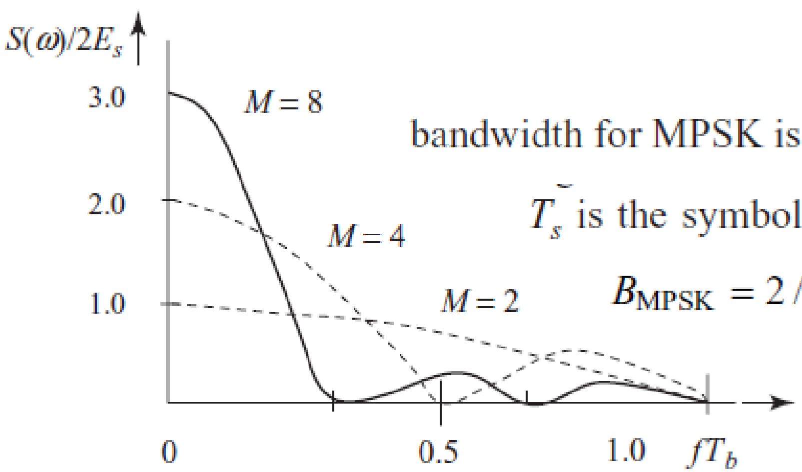
average error probability of symbol error for coherent MPSK

For M-ary PSK, the symbol duration $T_s = T_b \log_2 M$, where T_b is the bit duration.

As obtained in the BPSK $\quad S(\omega) = 2E_b \sin c^2(\omega T_b /2)$

Substituting, E_b by E_s and T_b by T_s, the PSD for MPSK becomes,

$$S_{MPSK}(\omega) = 2E_s \sin c^2(\omega T_s /2) = 2E_b \log_2 M \sin c^2(T_b f \log_2 M)$$



bandwidth for MPSK is $B_{\text{MPSK}} = 2/T_s$,

T_s is the symbol duration.

$$B_{\text{MPSK}} = 2/T_b \log_2 M = 2R_b / \log_2 M$$

$$R_b = 1/T_b,$$

Normalized power spectra for MPSK

As M increases , BW efficiency increases and also the error probability
So E_b/N_o ratio is adjusted

The channel efficiency ρ_{ch} is defined as
= Bit rate /Bandwidth

$$\rho_{ch} = R_b / B_{\text{MPSK}} = \log_2 M /2$$

Quadrature Phase shift Keying -QPSK

QPSK is the special case of MPSK with *M = 4, symbols can take four possible values (10, 00, 01, 11) each corresponds to 2 bits (dibits).*

For QPSK signal, the phase of the carrier can take only four equal spaced values, such as, pi /4, 3pi/4, 5pi /4 and 7pi /4.

The phase information is used only to provide for synchronization of the receiver to the transmitter.

The transmitted signal is given by

$$s_i(t) = \begin{cases} \sqrt{2\,E_s \,/\, T_s}\,\cos\,[2\pi f_c t + (2i-1)\,\pi\,/4], & \text{over } 0 \le t \le T_s \\ 0, & \text{Otherwise} \end{cases}$$

$$i = 1, 2, 3 \text{ and } 4$$

$$s_i(t) = \sqrt{2E_s\,/T_s}\,\cos\,[(2i-1)\,\pi\,/4]\cos 2\pi f_c t - \sqrt{2E_s\,/T_s}\,\sin\,[(2i-1)\,\pi\,/4]\sin 2\pi f_c t$$

In terms of basis functions,

$$s_i(t) = \sqrt{E_s}\cos[(2i-1)\pi/4]\,\varphi_1(t) - \sqrt{E_s}\sin[(2i-1)\pi/4]\,\varphi_2(t)$$
$$= s_{i1}\varphi_1(t) + s_{i2}\varphi_2(t)$$

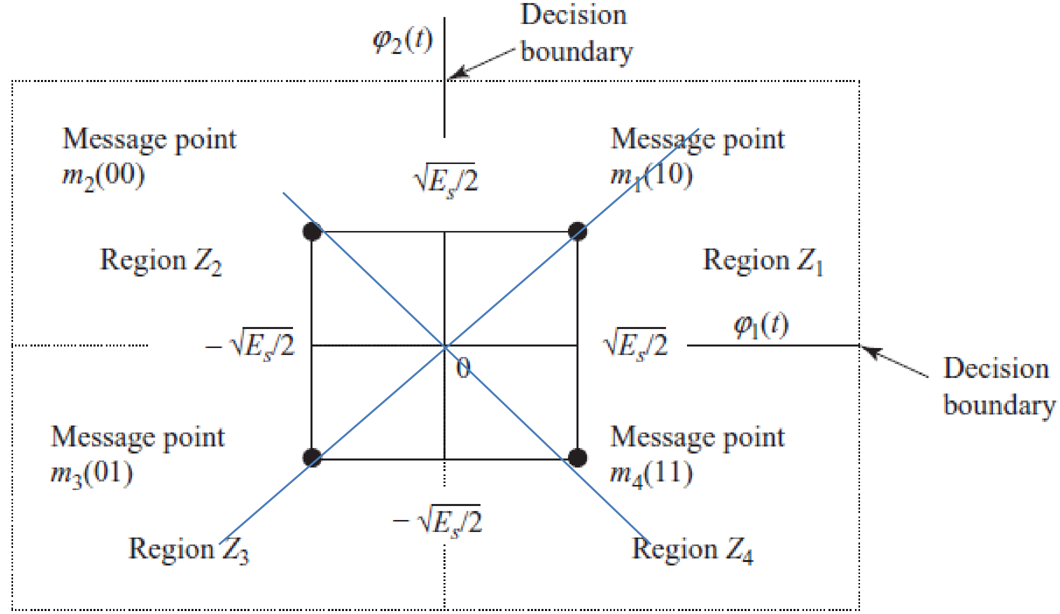
where

$$\varphi_1(t) = \sqrt{2/T_s}\cos 2\pi f_c t \qquad \varphi_2(t) = \sqrt{2/T_s}\sin 2\pi f_c t \qquad \text{for } 0 \le t \le T_s$$

two-dimensional signal space defined by the two-orthonormal sets $\{\varphi_1(t), \varphi_2(t)\}$,

There are four message points for QPSK signal that can be defined by signal vector as

	radians	s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E_s/2}$	$-\sqrt{E_s/2}$
00	$3\pi/4$	$-\sqrt{E_s/2}$	$-\sqrt{E_s/2}$
01	$5\pi/4$	$-\sqrt{E_s/2}$	$+\sqrt{E_s/2}$
11	$7\pi/4$	$+\sqrt{E_s/2}$	$+\sqrt{E_s/2}$



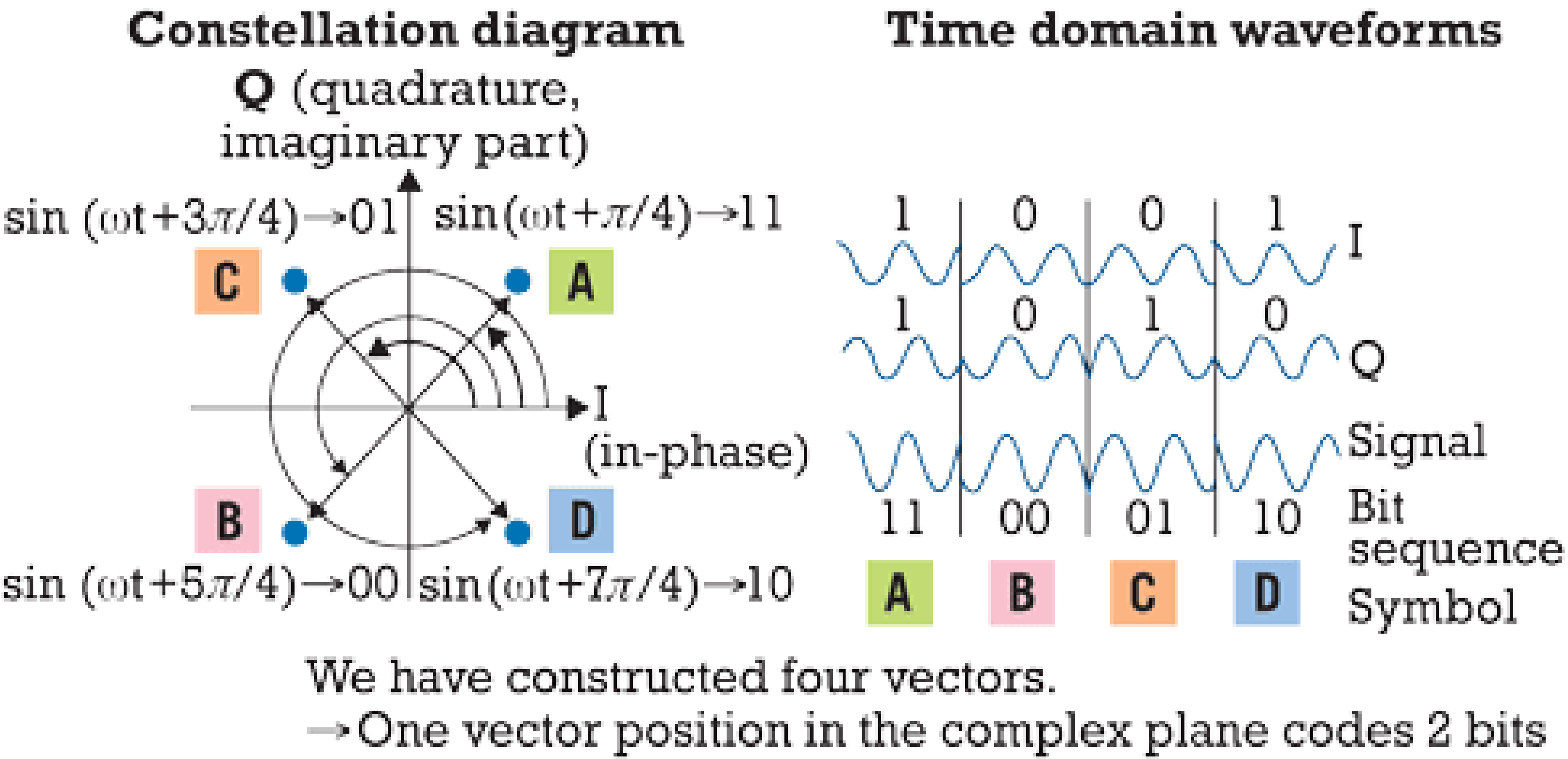
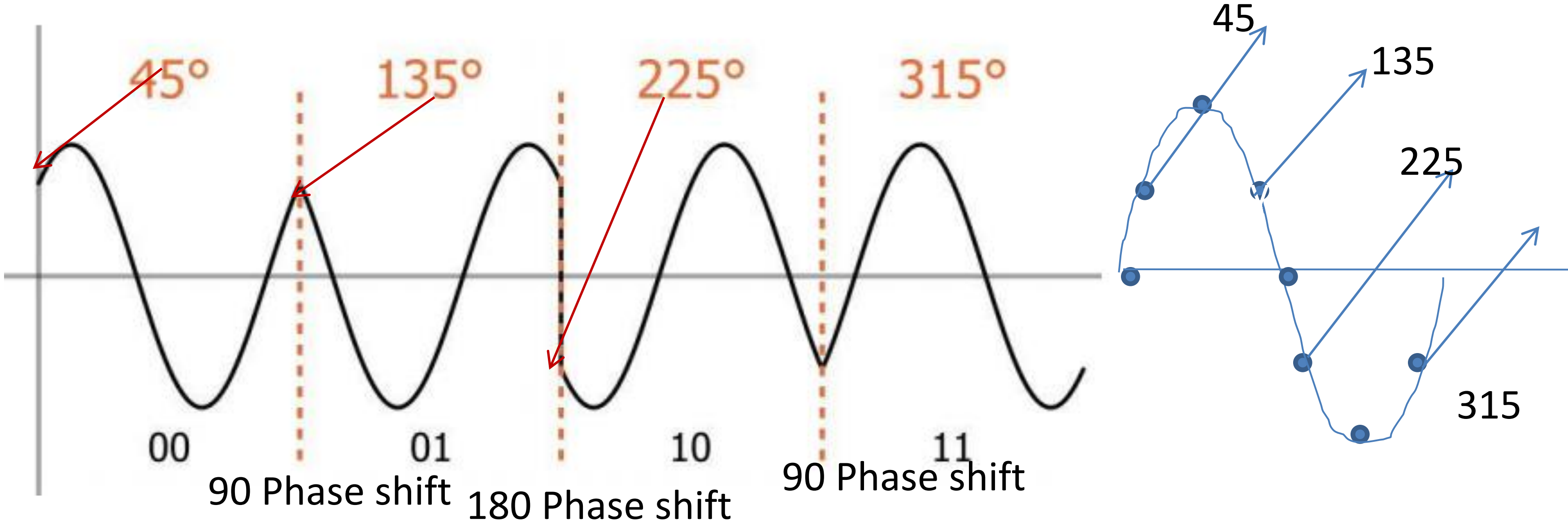


FIGURE 5. In QPSK, four symbols encode 2 bits each.

In QPSK, **the carrier varies in terms of phase**, not frequency, and there are four possible phase shifts



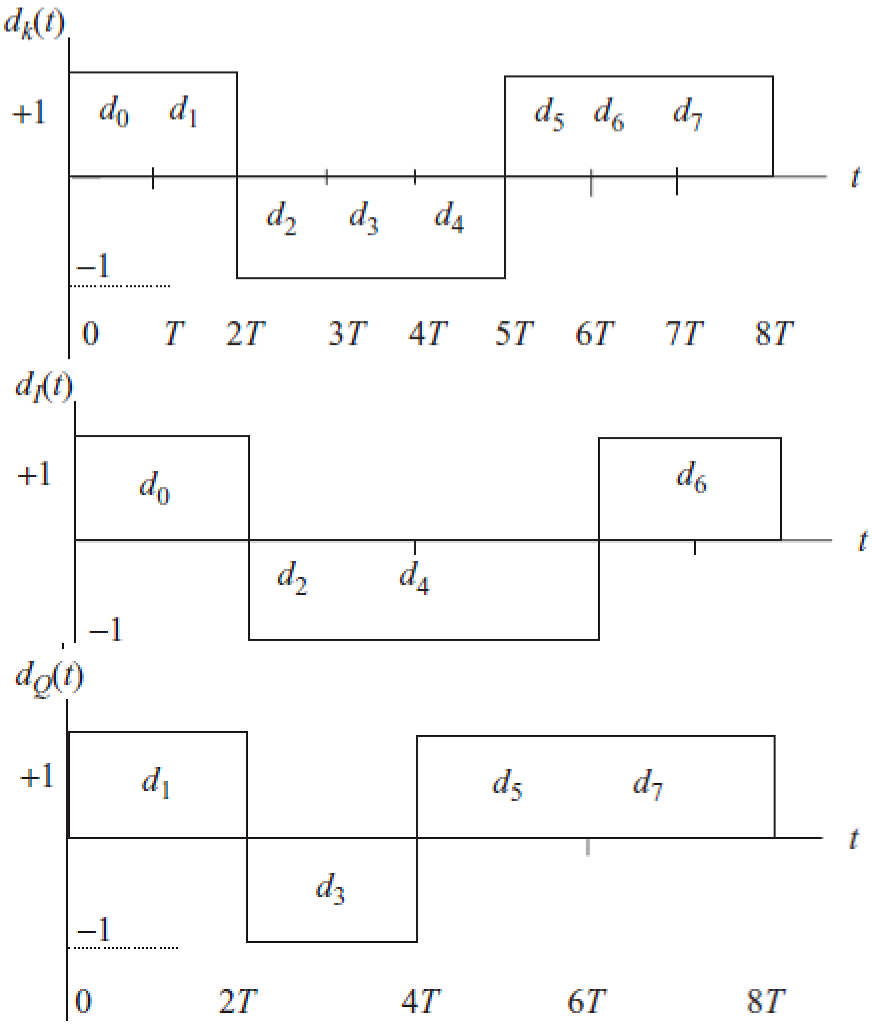
As long as the phase changes are gradual, amplitude variations in PSK are smooth and more or less maintain a constant envelope from one state to the other.

A 180° phase change, however, causes the envelope to invert. This abrupt change in the envelope, introduces spurious high frequency components in the QPSK spectrum.

Gradual phase change is more preferable. Hence Offset QPSK was devised which accounts for an intermediate 90° phase shift in time T as opposed to a 180° phase shift directly in time 2T.

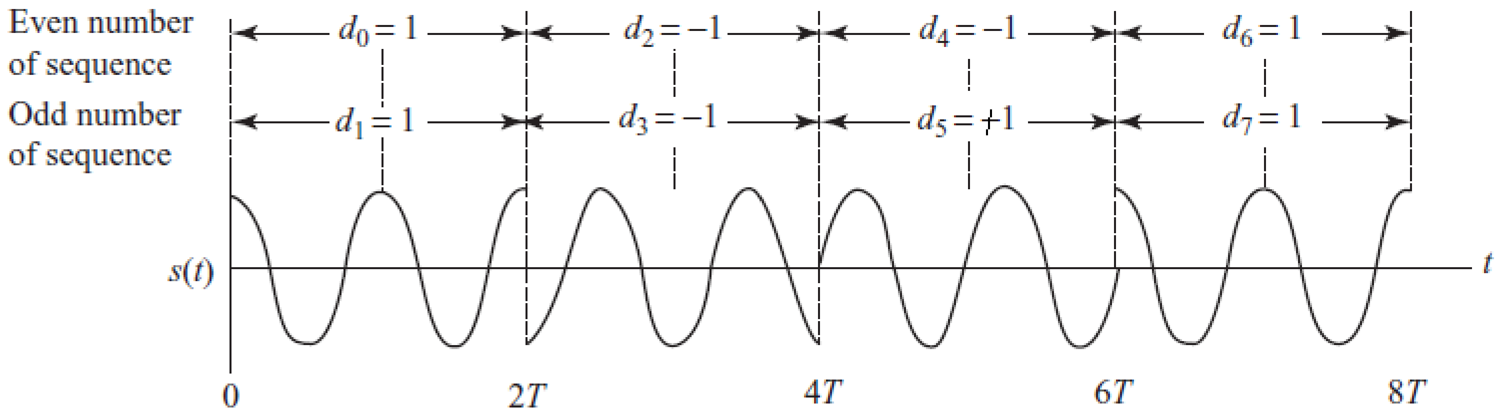
Each of the four possible phases of carriers represents two bits per symbol.

Since the symbol rate for QPSK is half the bit rate, twice as much data can be carried in the same amount of channel bandwidth as compared to BPSK.



This is possible because the two signals I (inphase) and Q (Quadrature) are orthogonal to each other and can be transmitted without interfering with each other.

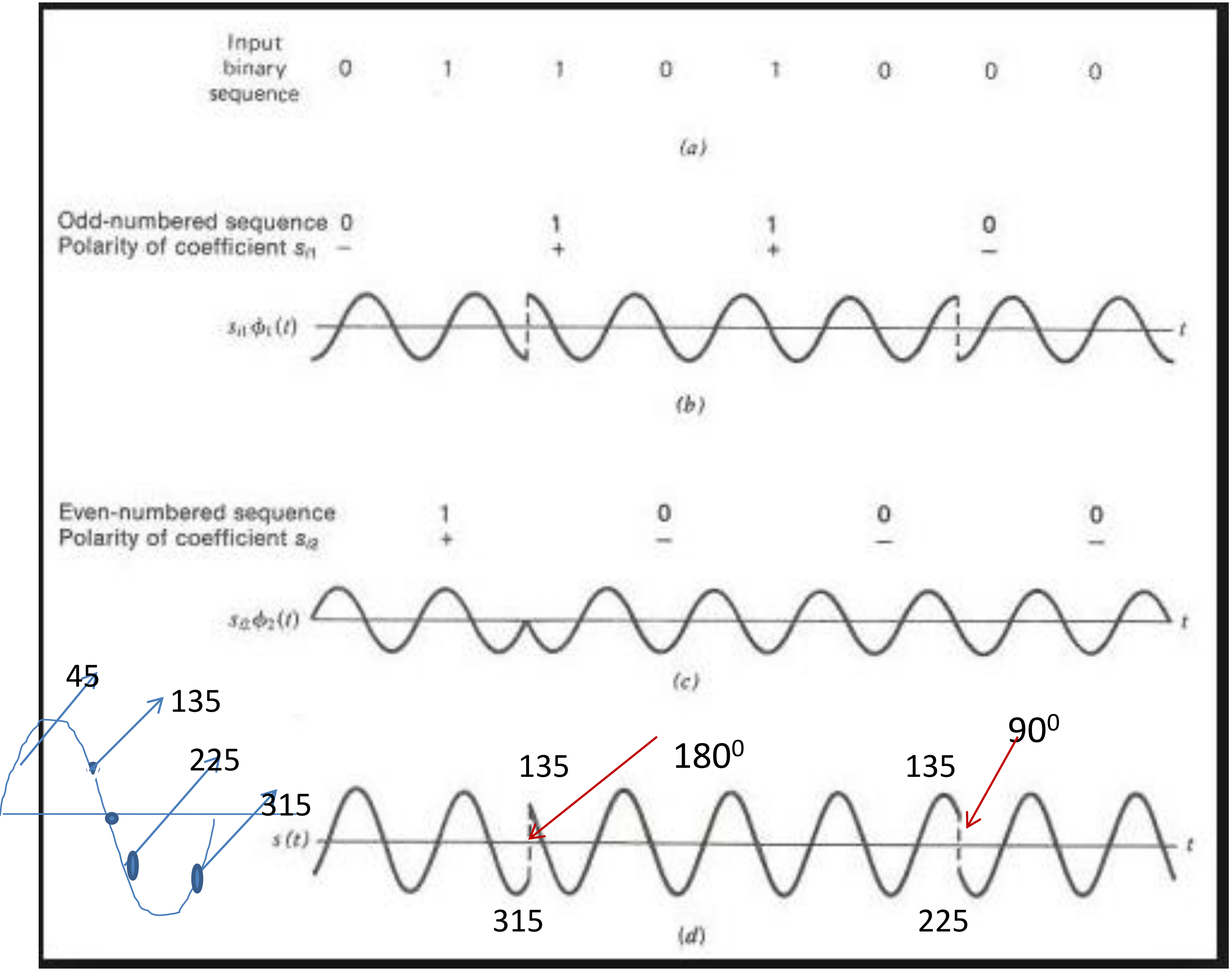
In QPSK the carrier phase can change only once every 2*T secs.* *If from one T interval to the next one, neither bit stream changes sign, the carrier phase remains unchanged.*



(b)

If one component d_I (t) or d_Q (t) changes sign, a phase change of pi/2 occurs (Example: when the binary sequence switches from dibit 10 to dibit 00).

However, if both components change sign then a phase shift of pi occurs (Example: when the binary sequence switches from dibit 01 to dibit 10).



Odd Comp :
0 means
inverted cosine

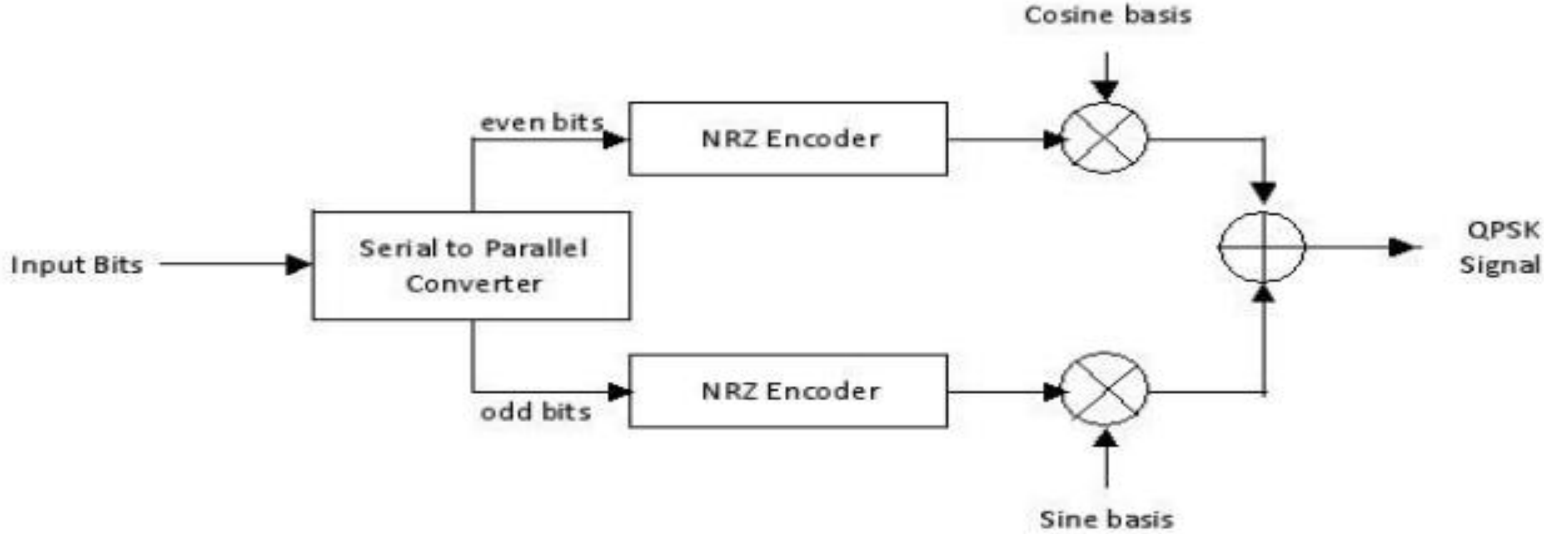
1 means cosine

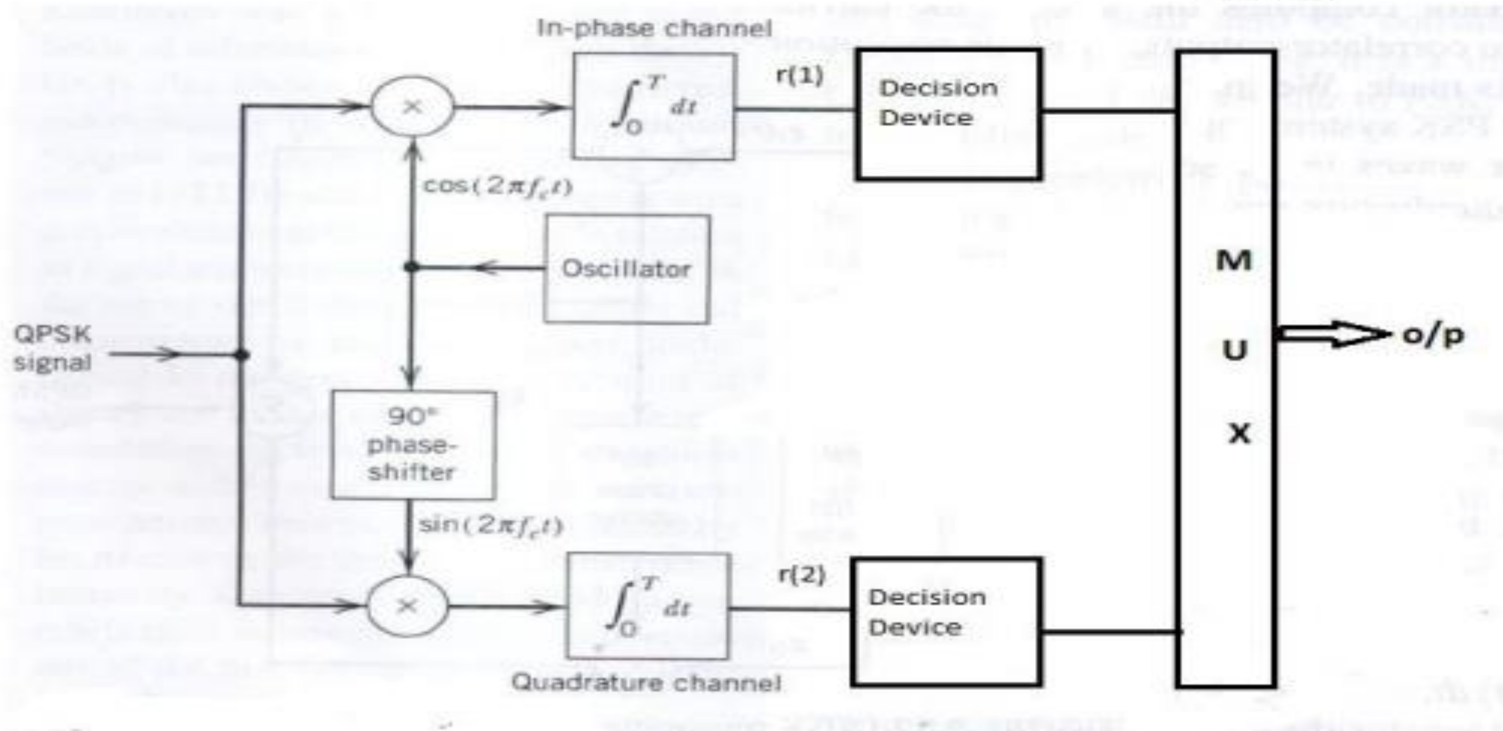
For even comp:
O means sine
1 means
inverted sine

Separate odd and even bits and generate waves with this consideration, run over 2T

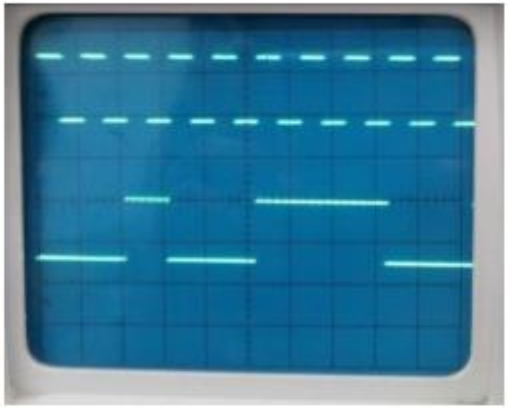
Addition of two BPSK is the QPSK

QPSK Modulation and Demodulation

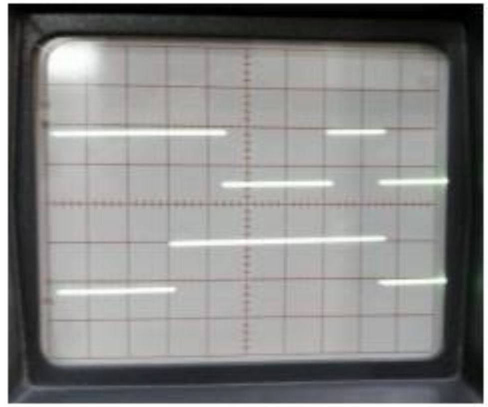




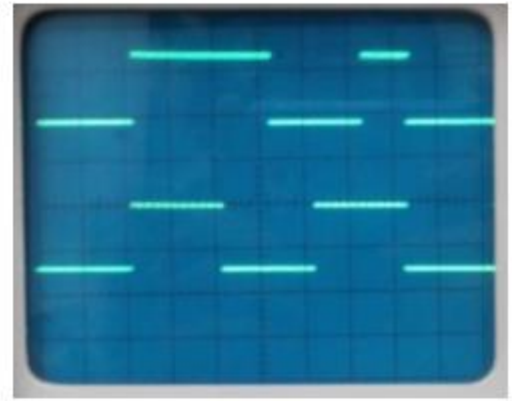
Waveforms on the Tx side:



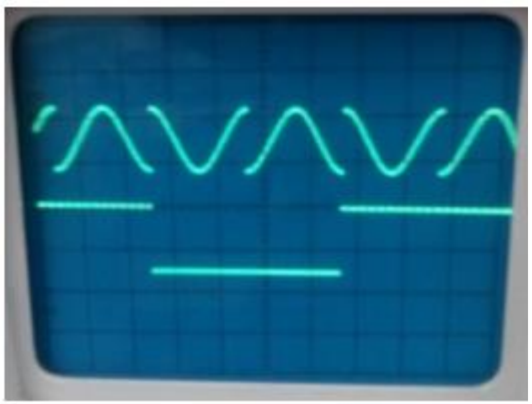
(a) Clock data vs generated data



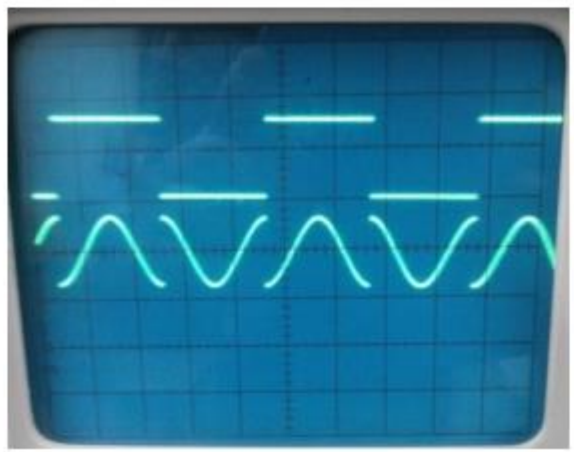
(b) Generated bitstream vs Even sequen



(c) Generated bitstream vs Odd sequence

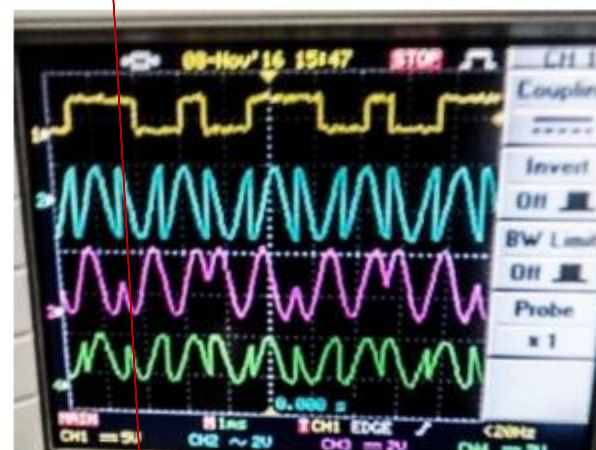


(d) Odd sequence vs BPSK1 (odd sequence m
Cosine basis function)



(e)Even sequence vs BPSK2

(multiplication with Sine of basis function)



(f)Generated bitstream vs BPSK1 vs BPSK2 vs QPSK

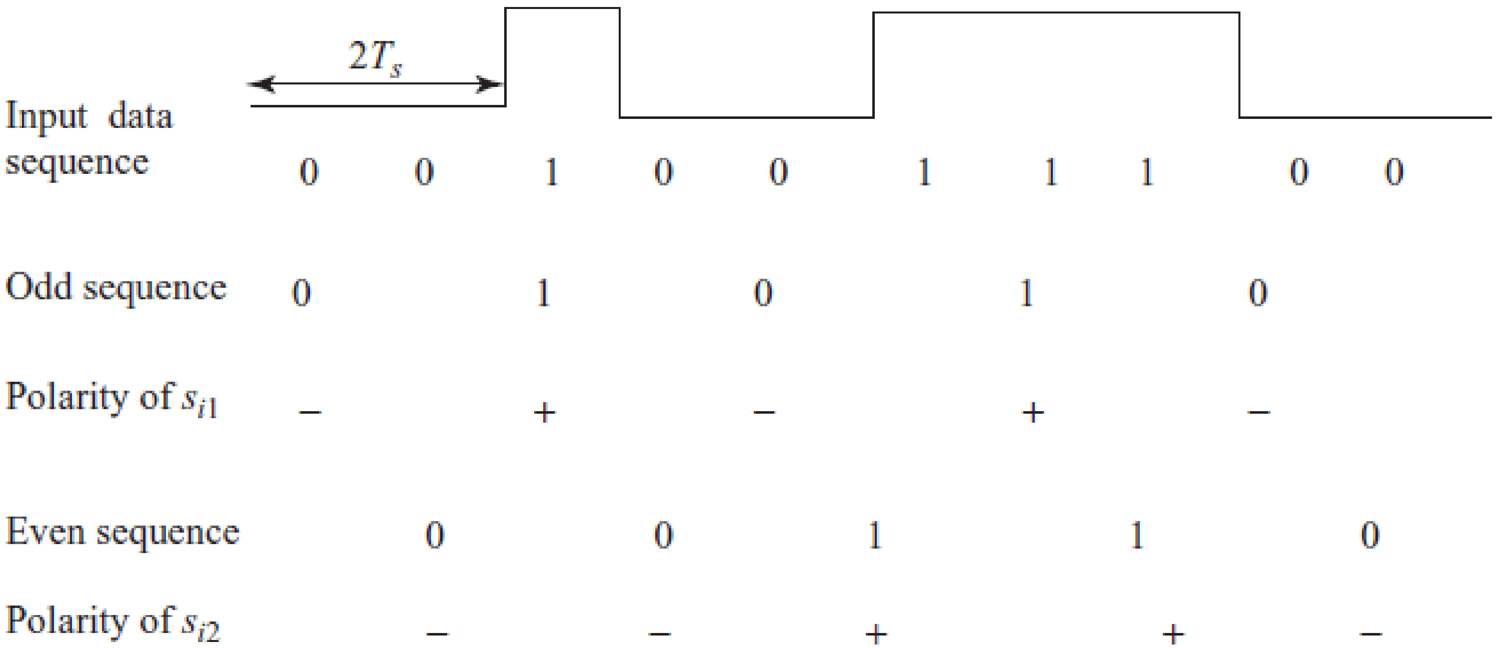
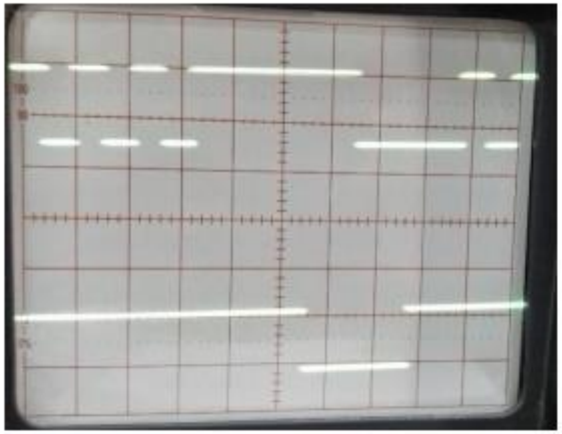


Fig. 5.25 *Odd-even sequences for QPSK signal showing the polarity*

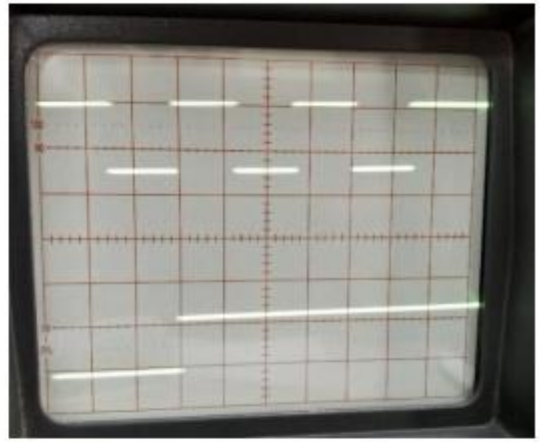
Table 5.2 *QPSK waveforms corresponding to dibits*

Input dibit	Nature of $s_{i1}\,\varphi_1(t)$	Nature of $s_{i2}\,\varphi_2(t)$
00	Inverted Cosine	Inverted Sine
10	Cosine	Inverted Sine
01	Inverted Cosine	Sine
11	Cosine	Sine
00	Inverted Cosine	Inverted Sine

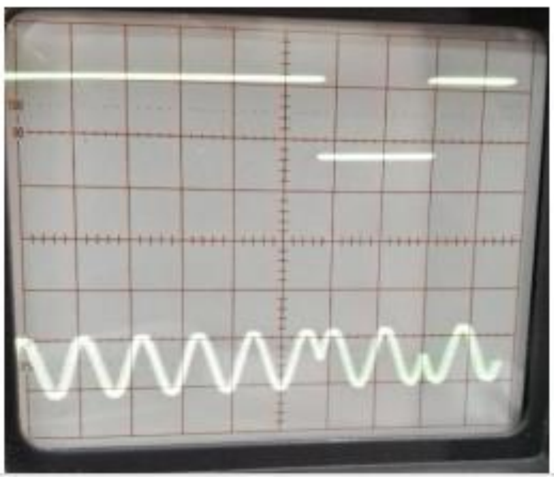
The simple algebraic summation of the in-phase *and quadrature waveforms generate the ultimate QPSK signal.*



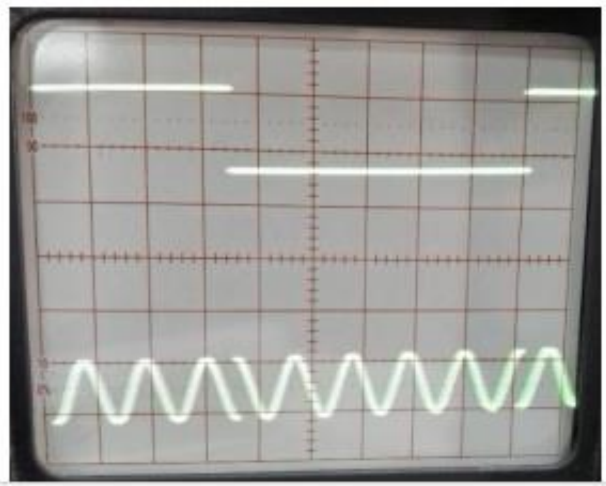
(a) Generated data vs even stream



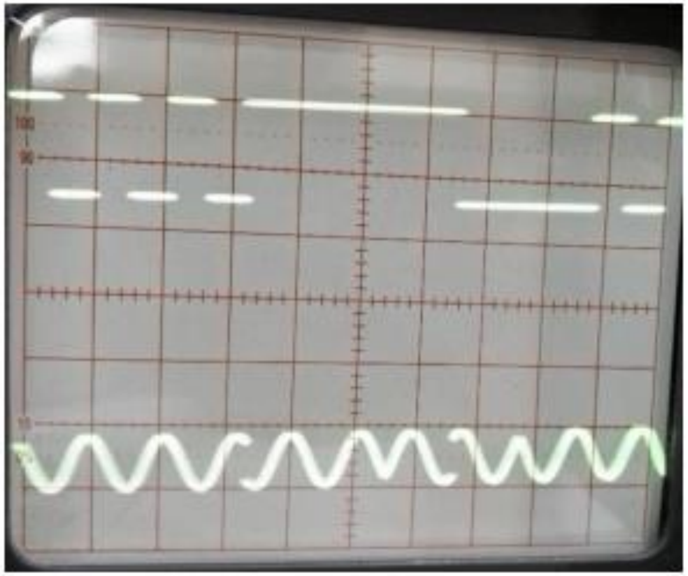
(b) Generated data vs odd stream



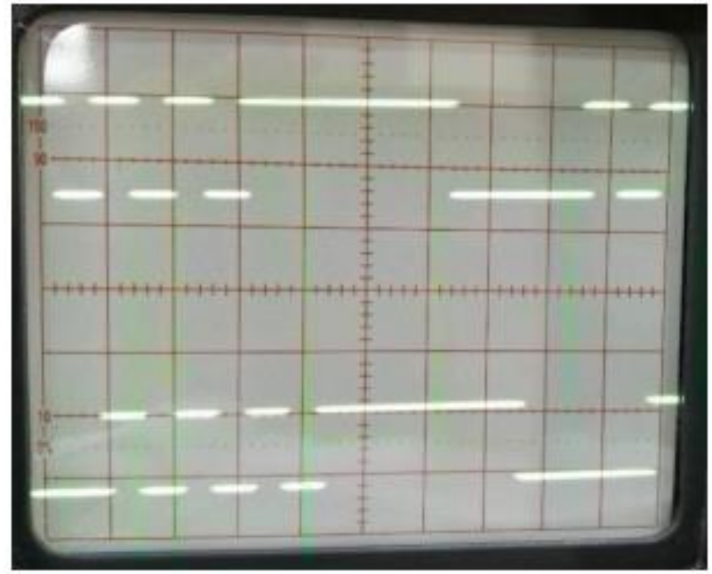
(c) Odd sequence vs BPSK1 (cosine multiplied waveform)



(d) Even sequence vs BPSK2(sine multiplied waveform)

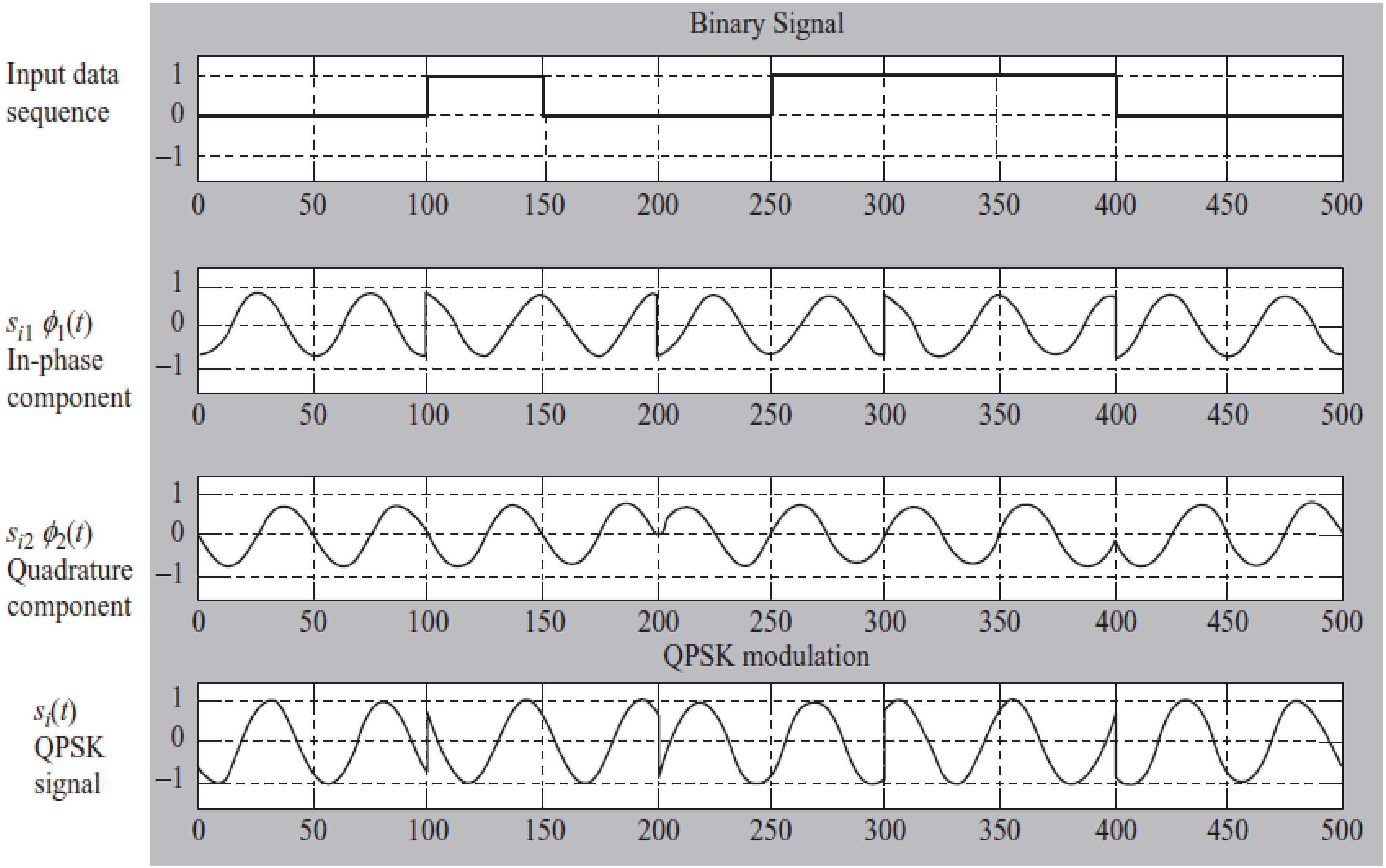


(e)Generated data vs QPSK modulated waveform

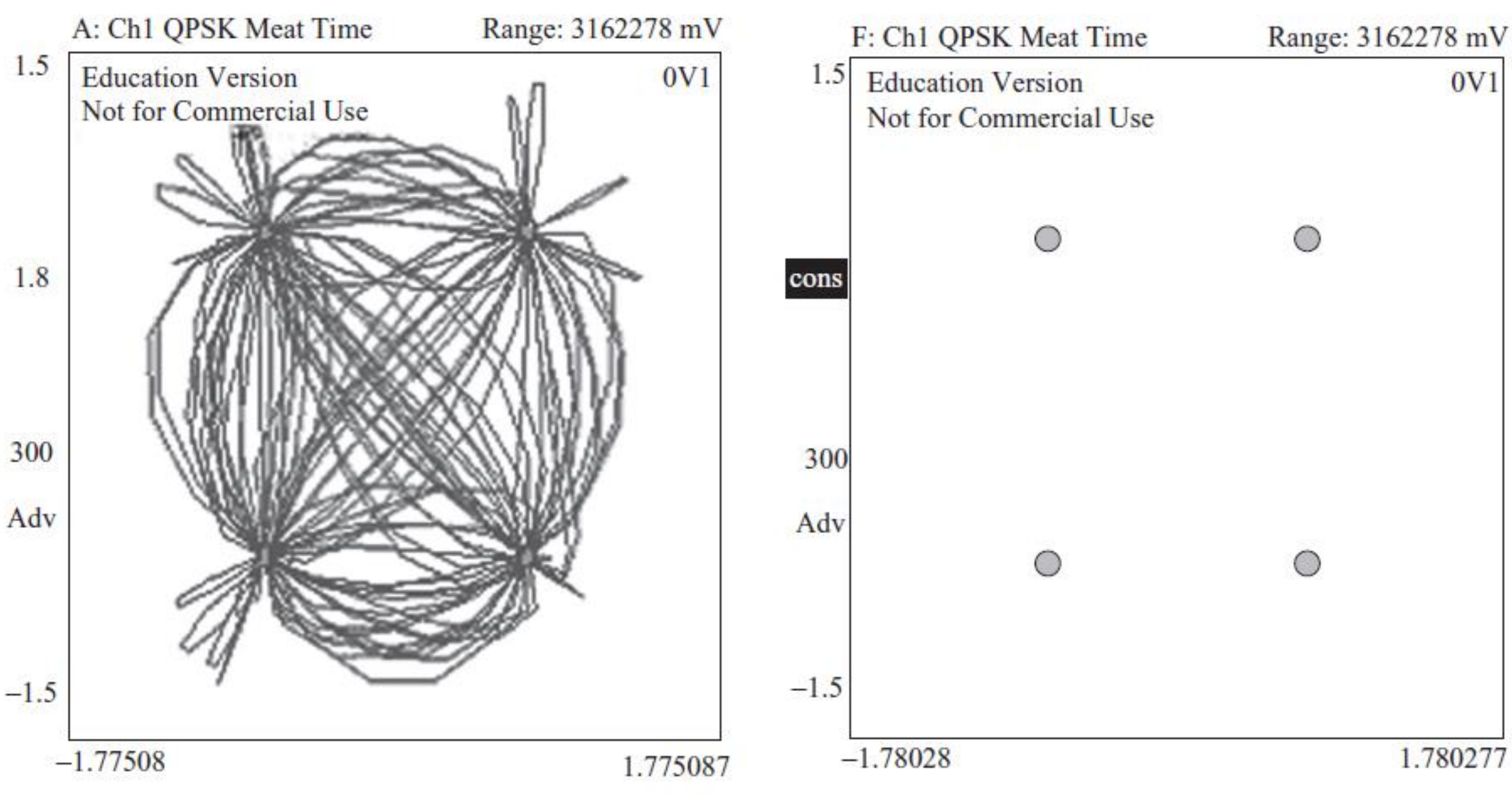


(f) Generated data vs QPSK demodulated waveform

Bit Pattern: 00 10 01 11 00



Experimental observation



Four message points in QPSK is circularly symmetric with respect to origin.

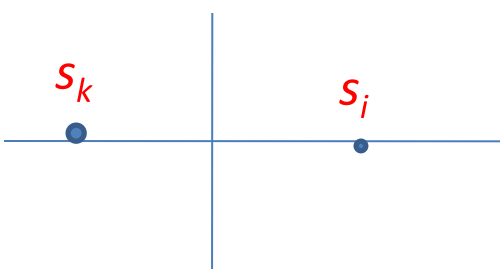
Error Probability of QPSK

Two equally likely messages are represented by two vectors s_i *and* s_k. *The decision boundary is represented by the bisector that is perpendicular to the line joining the points* s_i *and* s_k. *When the message* m_i *(vector* s_i*) is sent, and if the observation vector r lies on the side of the bisector where* s_k *lies an error is made. The probability of this event is*

$$P(s_i, s_k) = P\,(r \text{ is closer to } s_k \text{ than } s_i, \text{ when } s_i \text{ is sent})$$

$$= \int_{d_{ik}/2}^{\infty} 1/(\pi N_0)\exp(-v^2/N_0)\,dv$$

d_{ik} is the Euclidian distance between s_i and s_k.



$$Let\ z = v/\sqrt{N_0} \qquad \text{erfc}\,(u) = 2/\sqrt{\pi}\int_u^{\infty}\exp(-z^2)\,dz$$

$$P_e(m_i) \le 1/2 \sum_{\substack{k=1 \\ k\neq i}}^{M} \text{erfc}\,(d_{ik}/2\sqrt{N_0}) \quad i = 1,2,3.\ \&\ M$$

all the message points m_i

$$P(s_i, s_k) = 1/2\ \text{erfc}\,(d_{ik}/2\sqrt{N_0})$$

The probability of symbol error averaged over all the M message symbols is overbounded as

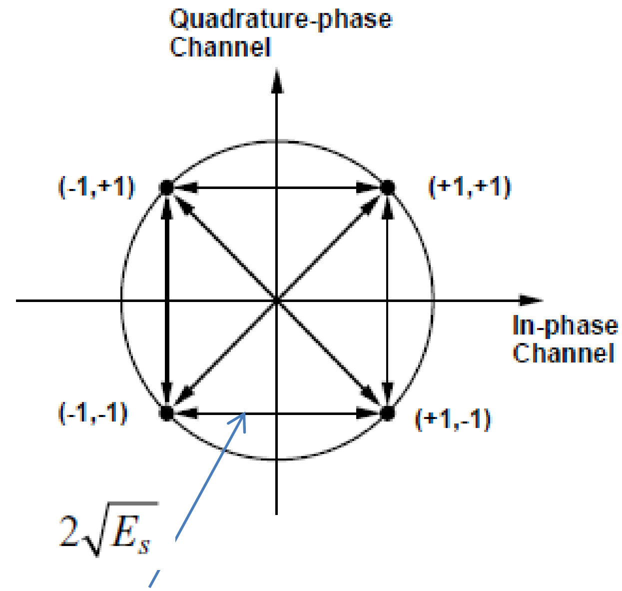
$$p_c = \sum_{i=1}^{M} p_i\, P_e(m_i) \le 1/2 \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \ne i}}^{M} p_i \operatorname{erfc}\left(d_{ik} / 2\sqrt{N_0}\right)$$

where p_i is the probability of transmitting symbol m_i.

Now consider the symbol transmission is circularly symmetric about the origin. Then the conditional probability P_e (m_i) *is same for all i, in that situation*

$$p_c \le 1/2 \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(d_{ik} / 2\sqrt{N_0}\right) \quad \text{for all } i$$

Now, considering QPSK signal, let the message point *m1 corresponding to dibit 10 is transmitted. The closest two message points are m2 and m4 corresponding to 00 and 11 and they are equidistant from m1.*



$$\text{So, } d_{12} = d_{14} = 2\sqrt{E_s}$$

The message point m_3 is at more distant point with large E_s/N_0 value.

From the observation vector in detecting message m_1, *mistakenly message* m_2 *or* m_4 *can be considered. In doing so, a single bit error may occur.*

For large E_s/N_0, *the chances of making two bits error is much less compared to 1 bit error. So, in calculating* p_e, m_3 *is excluded when message* m_1 *is sent.*

$$d_{ik} = d_{12} = d_{14} = 2\sqrt{E_s} \text{ and } E_s = 2E_b$$

The average probability of symbol error in each channel of a coherent QPSK system is $p_c = \mathrm{erfc}\left[\sqrt{E_b/N_0)}\right]$

Bit Error Rate (BER) for QPSK signals is

$$\text{BER} = 1/2 \text{ erfc } (\sqrt{E_b / N_0})$$

The in-phase and quadrature channels in QPSK system are statistically independent.

The bandwidth of QPSK signal is half of the bandwidth of the BPSK signal. But the noise immunity in both schemes is same for the same bit rate and same E_b / N_0

Power Spectra of QPSK Signals

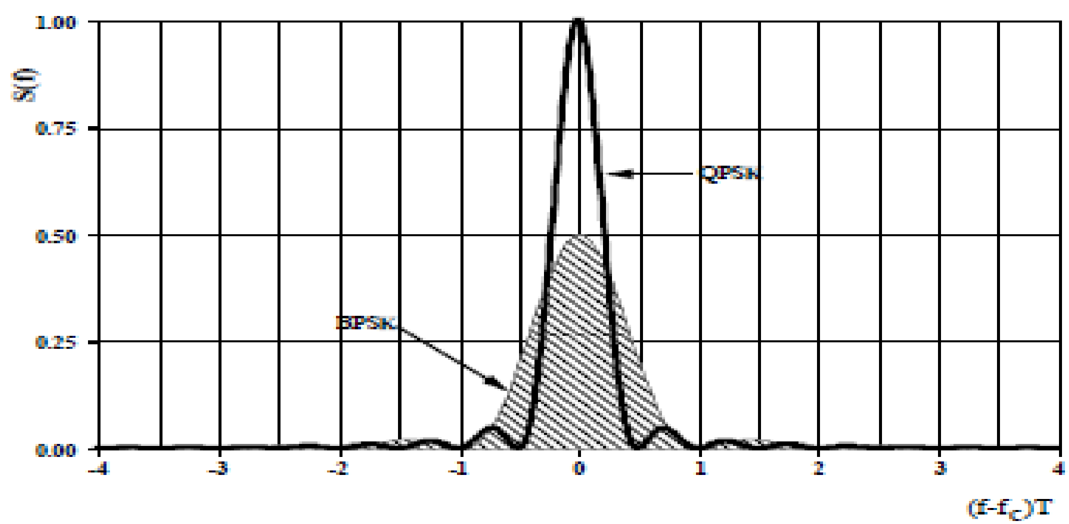
In QPSK, depending on the dibit sent during the signaling interval *Ts, the amplitude of in-phase component equals to + g (t) or – g (t), similarly for quadrature component. So, the symbol shaping function g (t) of the QPSK signal can be defined as*

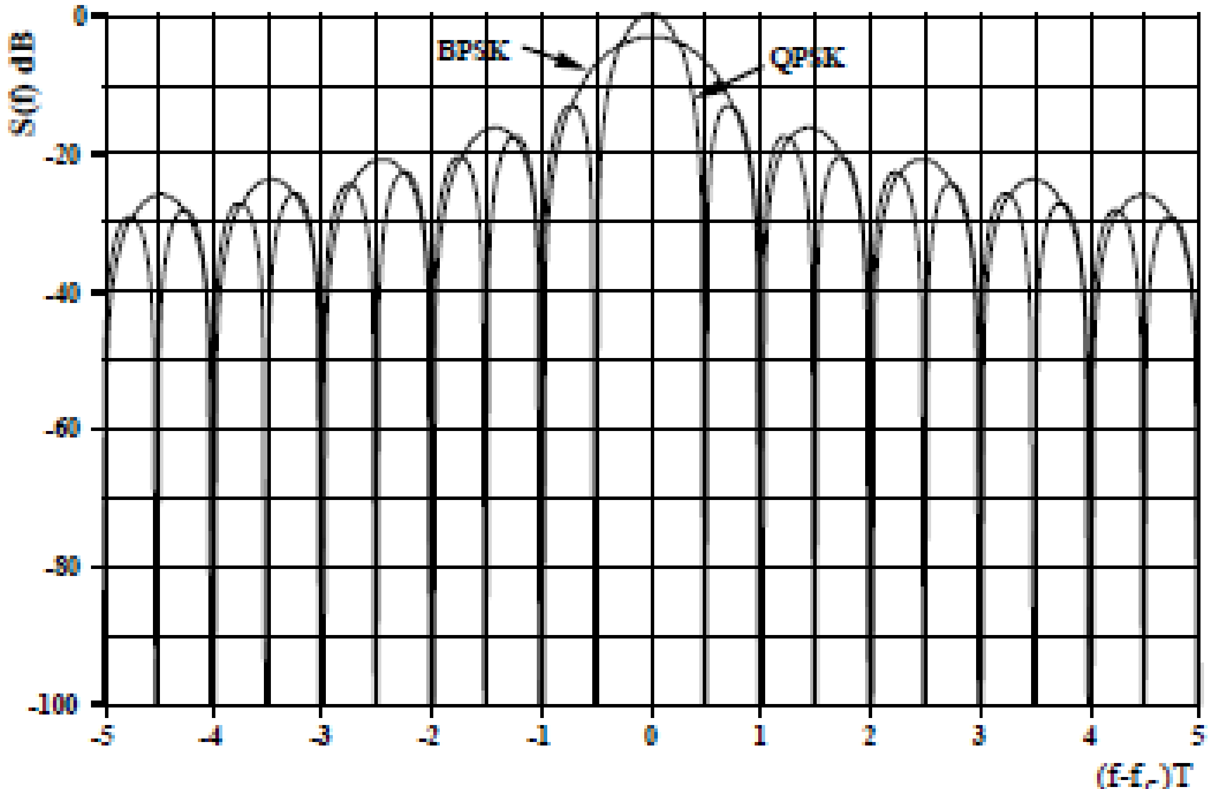
$$g(t) = \begin{cases} \sqrt{E_s / T_s} & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases}$$

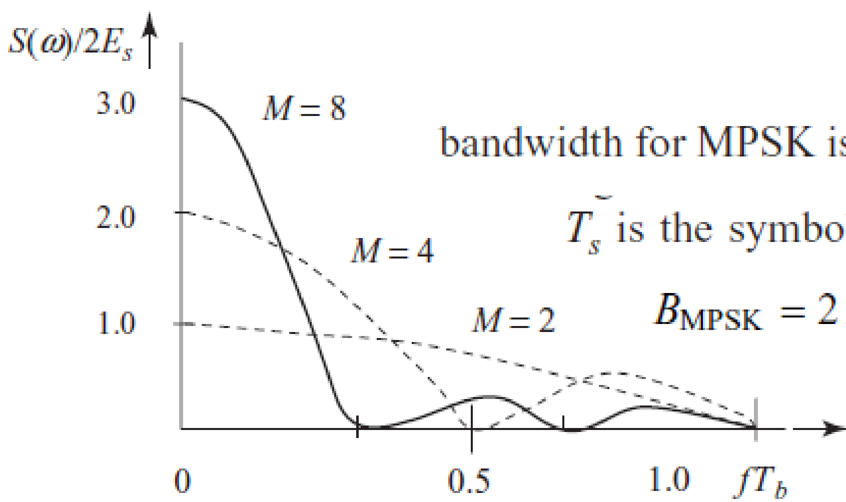
Assuming the binary wave at the modulator input is random and symbol 1 and 0 are equally likely and transmission at two adjacent time slots is statistically independent,
both the in-phase and quadrature components have common power spectral density each given as $E_s\, sinc^2\,(T_s f\,)$.

So, the power spectral density of QPSK signal is the sum of the two components as they are statistically independent components.

$$S_{\mathrm{QPSK}} = 2E_s \operatorname{sinc}^2 (T_s f) = 4\, E_b \operatorname{sinc}^2 (2T_b f)$$ Corresponds M=4







bandwidth for MPSK is $B_{\mathrm{MPSK}} = 2/T_s$,

T_s is the symbol duration.

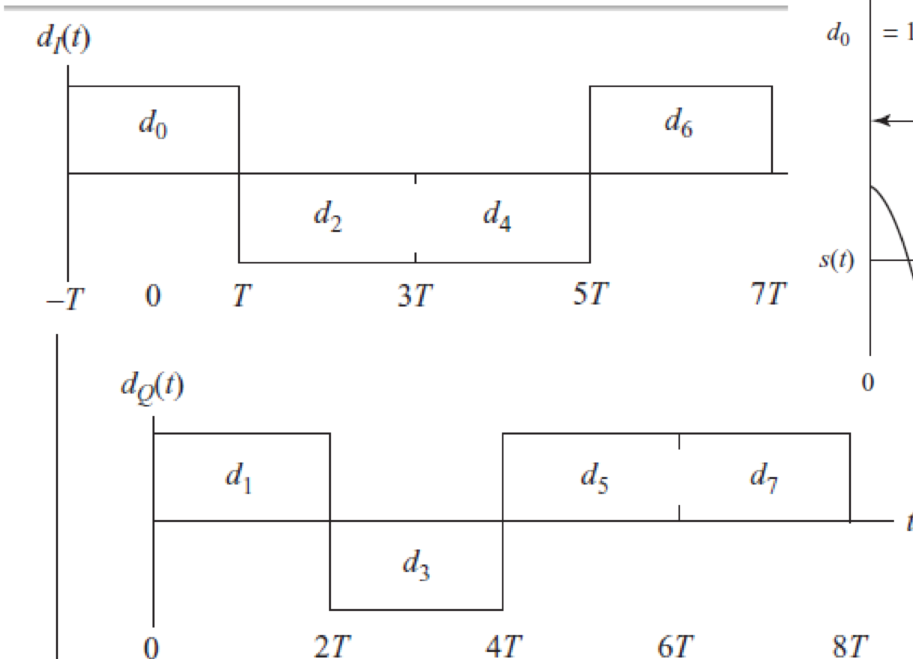
$$B_{\mathrm{MPSK}} = 2/T_b \log_2 M = 2R_b / \log_2 M$$

$$R_b = 1/T_b,$$

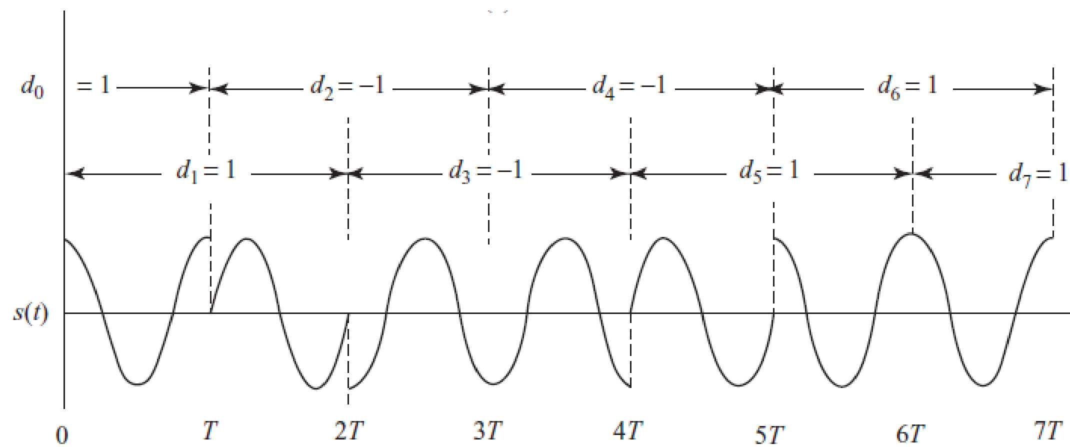
Offset Quadrature Phase shift Queuing (OQPSK)

➢If the two bit streams I and Q are offset by a bit interval, then the amplitude fluctuations are minimized since the phase never changes by 180° as it is occurred in QPSK when dibit 01 changes to 10.

➢This modulation scheme, Offset Quadrature Phase shift Keying (OQPSK) is obtained from QPSK by delaying the odd bit stream by half a bit interval with respect to the even bit stream



(a)



Coherent Frequency Shift Keying (**FSK**)

Coherent FSK is the non-linear method of pass band transmission. At first we shall discuss the binary FSK system then the Minimum Phase Shift Keying.

$$s_i(t) = \begin{cases} \sqrt{2E_b/T_b} \cos(2\pi f_i t) & i = 1,2\ ,\ 0 \le t \le T_b \\ 0, \text{ elsewhere} \end{cases}$$

$f_i = (n + i)/T_b$, for some fixed integer n.

$$\varphi_i(t) = \begin{cases} 2/T_b \cos(2\pi f_i t) & i = 1,2\ ,0 \le t \le T_b \\ 0, \text{ otherwise} \end{cases}$$

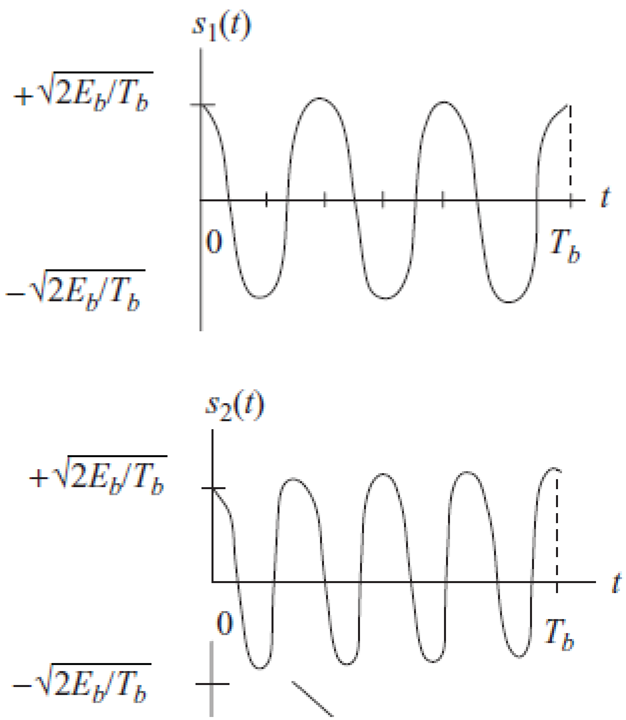
$$s_{ij} = \int_0^{T_b} s_i(t)\varphi_j(t)\,dt = \begin{cases} \sqrt{E_b} & i = j \\ 0 & i \ne j \end{cases}$$

The phase continuity of the signals is always maintained along with inter-bit switching times. This is known as *continuous phase frequency shift keying (CFSK).*

The coefficient s_{ij} for $i = 1,2$ and $j = 1,2$ is defined as, Unlike binary PSK, a binary FSK system is characterized by two-dimensional vector space (*N = 2) with two message points (M = 2) as*

The two message points are defined as $s_1 \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$ $s_2 \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$

The Euclidean distance between them is equal to $\sqrt{2E_b}$



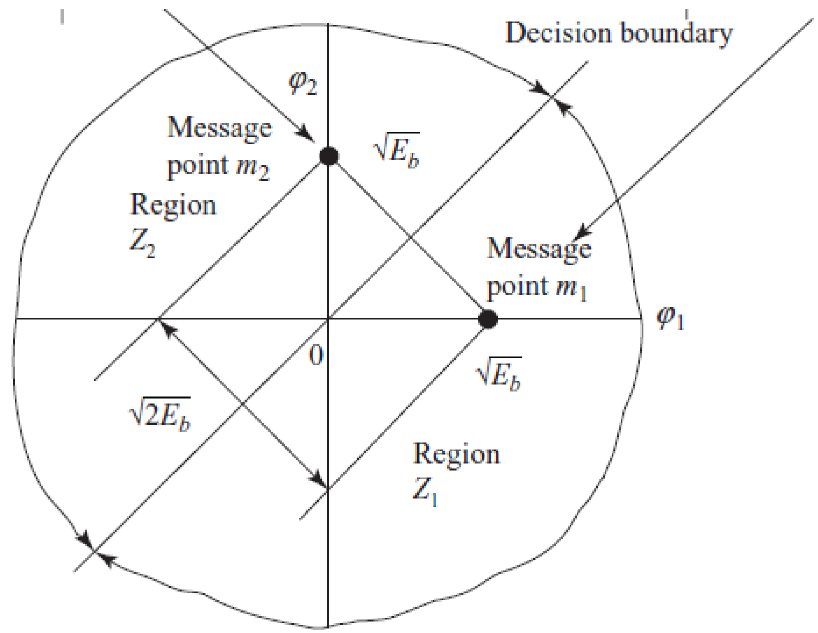


Fig. 5.37 *Signal-space diagram for BFSK signals*

Error Probability of BFSK Signals

The observation vector r has two elements

$$r_1 = \int_0^{T_b} r(t)\varphi_1(t)\,dt \qquad r_2 = \int_0^{T_b} r(t)\varphi_2(t)\,dt$$

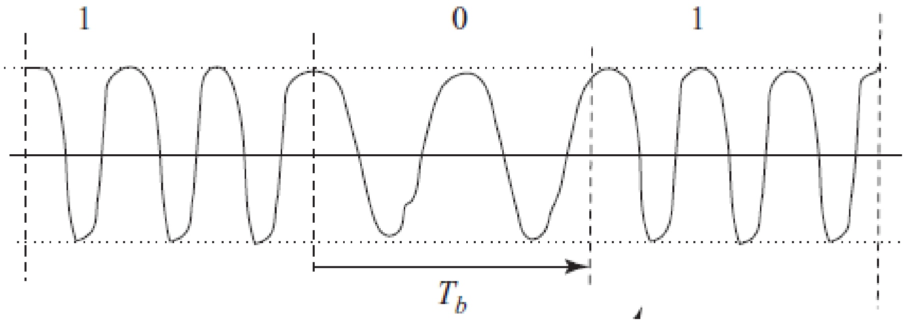
When symbol 1 is transmitted, r(t) equals $s_1(t) + w(t)$
0 is transmitted, $r(t)$ equals $s_2(t) + w(t)$.

The receiver decides in favour of symbol 1 if the received signal point represented by the observation vector *r falls inside region Z1, this occurs r1>r2 and if r1<r2, the received signal point falls inside region Z2, receiver decides in favour of symbol 0.*

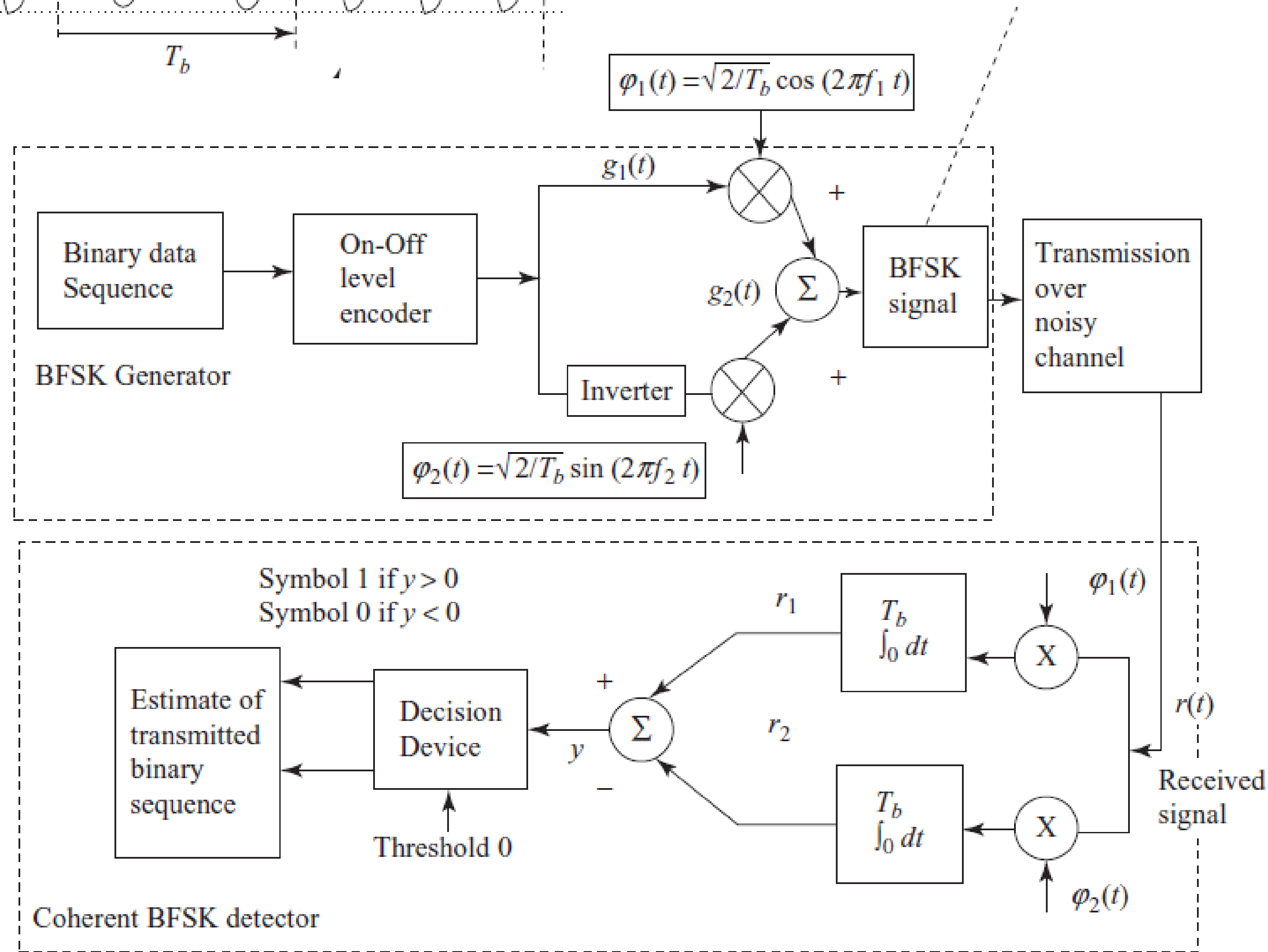
$$P(s_i, s_k) = P\ (r \text{ is closer to } s_k \text{ than } s_i, \text{ when } s_i \text{ is sent})$$

$$= 1/2\ \operatorname{erfc}\ (d_{ik}/2\sqrt{N_0}) \ = \ 1/2\ \operatorname{erfc}\ (\sqrt{E_b/2N_0}\ ) \qquad d_{ik} \text{ as } \sqrt{2E_b}$$

This is to be mentioned that for BPSK the Euclidean distance is $2\sqrt{E_b}$ and the error probability is $1/2\ \operatorname{erfc}\left(\sqrt{(E_b/N_0}\right)$ which is in perfect accordance with the two cases.



1
0
1
T_b



$\varphi_1(t) = \sqrt{2/T_b}\cos(2\pi f_1 t)$
$g_1(t)$
+
Binary data Sequence
On-Off level encoder
$g_2(t)$
Σ
BFSK signal
Transmission over noisy channel
BFSK Generator
Inverter
+
$\varphi_2(t) = \sqrt{2/T_b}\sin(2\pi f_2 t)$
Symbol 1 if $y > 0$
Symbol 0 if $y < 0$
r_1
$\int_0^{T_b} dt$
X
$\varphi_1(t)$
+
Estimate of transmitted binary sequence
Decision Device
Σ
y
r_2
$r(t)$
−
Received signal
$\int_0^{T_b} dt$
X
Threshold 0
$\varphi_2(t)$
Coherent BFSK detector

Power Spectra of BFSK signal

In Sunde's FSK, the transmitted frequencies *f1 and f2 differ by an amount equal to the bit rate* $1/T_b$ *and their arithmetic mean is equal to the carrier frequency* f_c*. Phase continuity is always maintained. The binary FSK signal can be expressed as,*

$$\begin{aligned} s(t) &= \sqrt{2E_b/T_b}\cos\left(2\pi f_c t \pm \pi t/T_b\right), \qquad 0 \le t \le T_b \\ &= \sqrt{2E_b/T_b}\ \{\cos(2\pi f_c t)\cos(\pm\pi t/T_b) - \sin(2\pi f_c t)\sin(\pm\pi t/T_b)\} \\ &= \sqrt{2E_b/T_b}\ \{\cos(2\pi f_c t)\cos(\pi t/T_b) - \sin(2\pi f_c t)\sin(\pi t/T_b)\} \end{aligned}$$

9), + sign is for transmitting symbol 0 and – sign is for transmitting symbol 1.

$$g(t) = \begin{cases} \sqrt{2E_b/T_b} & ,0 \le t \le T_b \\ 0 & \text{otherwise} \end{cases}$$

$$s(t) = \sum_{k=-\infty}^{\infty} g(t-kT_b)\,\mathrm{Re}\{\exp j(2\pi f_c t + I_k \pi t/T_b)\}$$

$$s(t) = \sum_{k=-\infty}^{\infty} g(t-kT_b)\ \left\{e^{jI_k \pi t/T_b}\right\} e^{j2\pi f_c t}$$

$$= \{g_I(t) + j g_Q(t)\}\, e^{j2\pi f_c t}$$

where $g_I(t)$ = in-phase component

$$= \sum_{k=-\infty}^{\infty} g(t-kT_b)\cos(\pi t/T_b) = \sqrt{2E_b/T_b}\cos \pi t/T_b$$

Which is completely independent of input binary wave.

$g_Q(t)$ = quadrature component

$$= \sum_{k=-\infty}^{\infty} I_k g(t-kT_b)\sin(\pi t/T_b) = \pm\sqrt{2E_b/T_b}\ \sin \pi t/T_b$$

The quadrature component is directly related with input wave. Within the symbol interval $0 \le t \le T_b$, *it is equal to –g (t) for symbol transmission 1 and +g (t) for symbol 0.*

Because of the randomness we consider ACF

The autocorrelation function of $g_I(t)$

$$R_{gIgI}(\tau) = (1/T_b)\int_0^{T_b} (2E_b/T_b)\cos(\pi(t+\tau/T_b)\cos(\pi t/T_b) = (E_b/T_b)\cos(\pi t/T_b)$$

The power spectrum of this component is

$$S_{gI}(f) = \text{Fourier transform of } R_{gIgI}(\tau) = \int_0^{T_b} (E_b/T_b)\cos(\pi t + T_b)\, e^{-j2\pi ft}\, dt$$

$$= (E_b/2T_b)\ [\delta(f - 1/2T_b) + \delta(f + 1/2T_b)\ ]$$

The Fourier transform of the $G_Q(t) = G_Q(f) = \int_0^{T_b} \sqrt{2E_b/T_b}\ \sin(\pi t/T_b) e^{-j2\pi ft}\, dt = \mathrm{I}$

Integrating by parts with the limit,

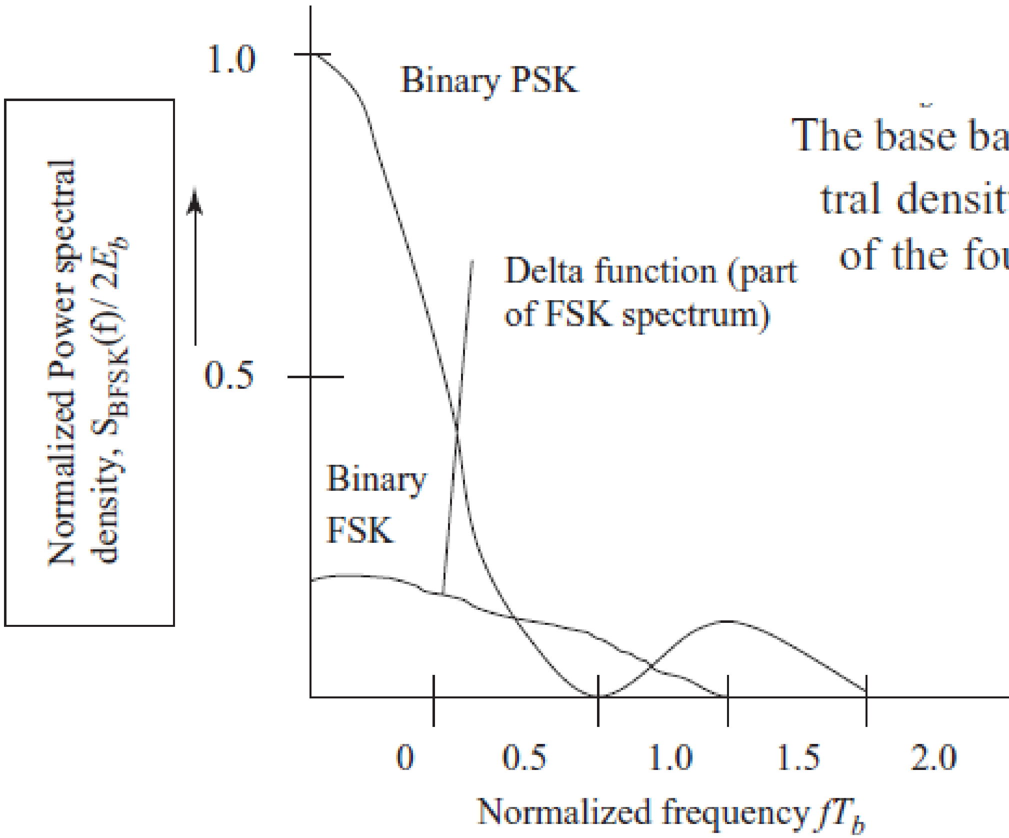
$$G_Q(f) = \frac{2\sqrt{2E_b/T_b}}{\pi}\ [\cos(\pi f T_b) / \ 1 - 4T_b^2 f^2]\ e^{(-j2\pi ft)}$$

$$S_{gQ}(f) = (1/T_b)\ G_Q(f) G_Q(-f)\ = \frac{4 \text{ x} (2E_b)}{\pi^2}\quad [\cos((\pi f T_b) / 1 - 4T_b^2 f^2]^2$$

Therefore, the power spectra for BFSK is the sum of the $S_{gI}(f)$ and $S_{gQ}(f)$

$$S_{\mathrm{BFSK}}(f) = (E_b/2T_b)\ [\delta(f - 1/2T_b)\ + \delta(f + 1/2T_b)] + (8E_b/\pi^2)\ [\cos((\pi f T_b) / 1 - 4T_b^2 f^2]^2$$

binary FSK has two frequency components $f_1 = f + 1/2T_b$ and $f_2 = f - 1/2T_b$,



The base band power spectral density of BFSK falls of as inverse of the fourth power of frequency f. faster roll off occurs.

If *f_1 and f_2 are not multiple of $1/T_b$,* BFSK is not continuous phase *then spectral roll off (rate of decay) is slower, in fact it decays as $1/f^2$ than that of $1/f^4$ as in continuous phase BFSK.*

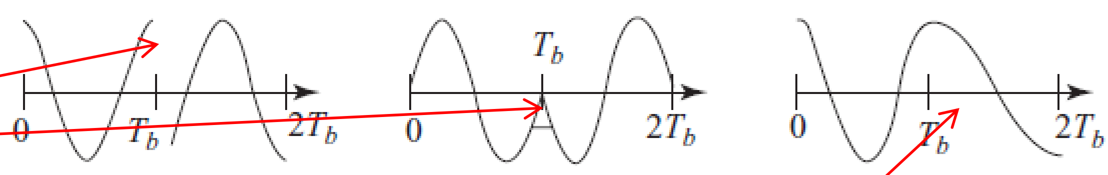
Minimum Shift Keying –MSK Why?

MSK is a special type of CPFSK, where the frequency changes occur at the zero crossings of the carrier. The modulation index for MSK is 0.5.

The modulation index of 0.5 corresponds to the minimum frequency spacing that makes two FSK signals to be coherently orthogonal and the name minimum shift keying implies the minimum frequency separation, i.e., bandwidth that allows orthogonal detection.

It is found that binary data has sharp transition between "1" and "0" states and vice versa. This creates the signal having side bands extending out of the carrier signal and causes interference to adjacent channels. By using filter, this problem can be partly overcome.

Causes spectral spread



phase-continuous signals in general have better spectral properties

MSK has one of two possible frequencies over any symbol interval. In traditional FSK, we use signals of two different frequencies of *f1 and f2 to transmit message s = 0 and message s = 1over a time* T_b.

$$s_1(t) = \sqrt{2E_b / T_b} \cos(2\pi f_1 t), \quad 0 \le t \le T_b$$

$$s_2(t) = \sqrt{2E_b / T_b} \cos(2\pi f_2 t), \quad 0 \le t \le T_b$$

We assume $f_1 > f_2 > 0$. Choice of frequencies are such that in each time interval T_b, there is an integer number of periods, $f_1 = k_1/T_b$ and $f_2 = k_2/T_b$ with k_1 and k_2 being integers, the signal is definitely continuous phase

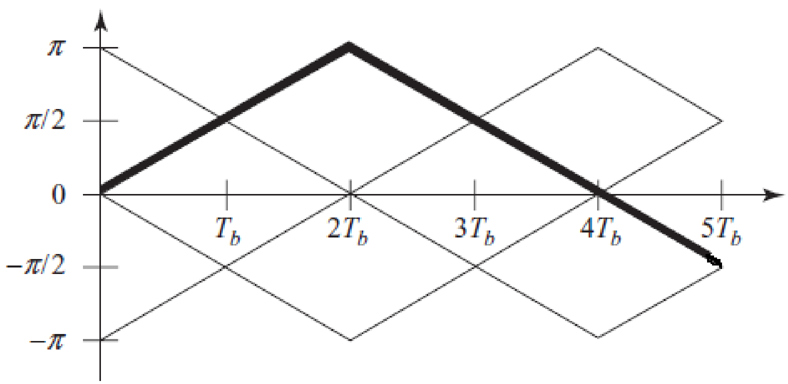
If either *f_1 or f_2 are chosen such that there is non-integer number of periods, the traditional FSK modulator will output a signal with significant discontinuities in the phase. In order to maintain the phase continuity, we can let the transmitter to have memory. We choose the signals for a general CPFSK as*

$$s(t)=\begin{cases}\sqrt{2E_b/T_b}\ \cos\,[\,2\pi f_1 t+\theta(0)]\ \text{ for symbol }\ 1=s_1(t)\quad 0\le t\le T_b\\ \sqrt{2E_b/T_b}\ \cos\,[\,2\pi f_2 t+\theta(0)]\ \text{ for symbol }\ 0=s_2(t)\quad 0\le t\le T_b\end{cases}$$

the phase continuity is kept by letting θ (0) be equal to the argument of the cosine pulse for the previous bit interval. For the signals over an arbitrary bit interval *$kT_b \le t \le (k+1)T_b$, the general phase memory term is $\theta(kT_b)$*

We assume modulation index *$h = 0.5$ and $\theta(0) = 0$ or $\theta(0) = \pi$.*

For every multiple of the bit time the phase can only take on one of two values, the values being 0 and π for $t = 2kT_b$ and ± pi/2 for $t = (2k+1)T_b$.



Phase trellis with h = 0.5, sequence 11000

Thus CPFSK with deviation ratio h = 0.5 is called MSK.

The frequency difference $f_d = f_1 - f_2 = 1/2T_b$ that results from choosing h = 0.5 is the smallest possible difference if the signals of the two frequencies are to be orthogonal over one bit interval.

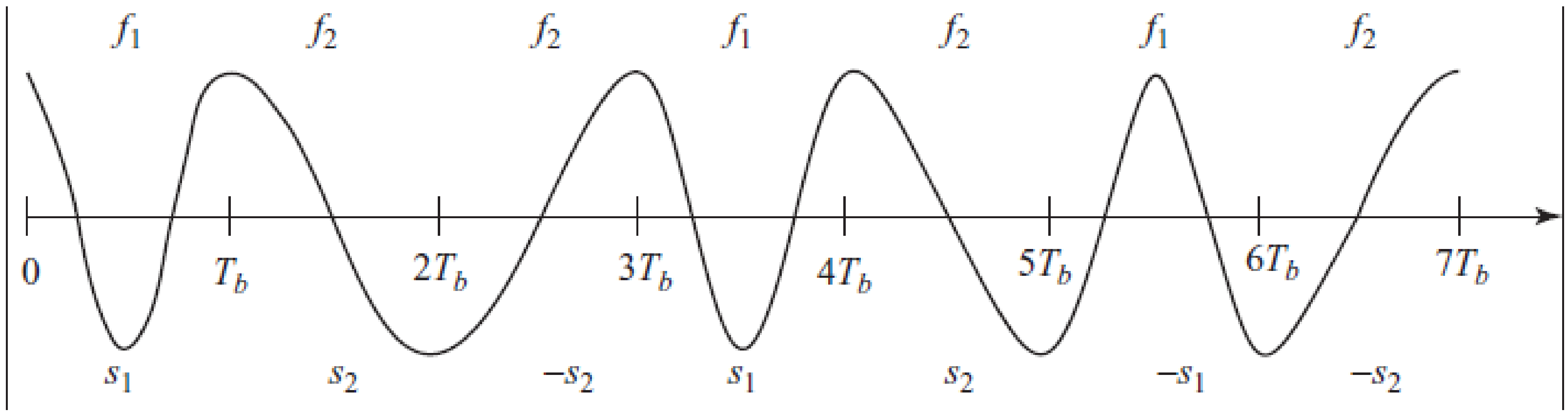


Fig. 5.42 *Example of MSK signal (CPFSK)*

MSK is unique due to the relationship between the frequencies of logic 0 and 1. The difference between the frequencies is always 1/2 the data rate. This is the minimum frequency spacing that allows 2 FSK signals to be coherently orthogonal that is why the name minimum shift keying.

In general the angle modulated wave for MSK is represented as

$$s(t) = \sqrt{2E_b/T_b}\ \cos\ [2\pi f_c t + \theta(t)]$$ The carrier frequency $f_c = ½\ (f_1 + f_2)$

$$\theta(t) = \theta(0) \pm \pi h t / T_b \qquad 0 \le t \le T_b$$

$$h = T_b(f_1 - f_2) = T_b f_d \qquad \theta(t) = \theta(0) \pm \pi f_d t$$

$$\theta(t) - \theta(0) = \begin{cases} \pi h & \text{for symbol 1} \\ -\pi h & \text{for symbol 0} \end{cases}$$

At time t = T_b , txmitting 1 increases CPFSK phase by πh 0 decreases phase by -

The + sign corresponds to the frequency of the carrier being shifted to higher frequency *$f_2 = f_c + 1/4T_b$ while for – sign the frequency of carrier f_c shifts to lower frequency $f_1 = f_c - 1/4T_b$. So*

$$s(t) = \sqrt{2E_b / T_b}\cos(2\pi f_c t)\cos(\theta(t)) - \sqrt{2E_b / T_b}\,\sin(2pf_c t\,)\sin(\theta(t))$$

$$= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(\,2\pi f_c t\,)$$

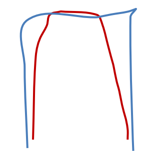
With $h = 0.5\ \theta(t)$

$$\theta(0) \pm \pi f_d t = \theta(0) \pm \pi t / 2T_b \text{ over } 0 \le t \le T_b.$$

The plus sign corresponds to symbol "1" and the minus sign corresponds to symbol "0".

$$s_I(t) = \sqrt{2E_b / T_b}\ \cos(\theta(t)) = \sqrt{2E_b / T_b}\ \cos(\theta(0))\ \cos\ (\pi t / 2T_b)$$

$$= \pm\sqrt{2E_b / T_b}\ \cos(\pi t / 2T_b), \qquad T_b \le t \le T_b$$



Where the + sign corresponds to $\theta(0) = 0$ and the minus sign corresponds to $\theta(0) = \pi$.

$$s_Q(t) = \sqrt{2E_b / T_b}\ \sin(\theta(t)) = \sqrt{2E_b / T_b}\ \sin\ ((\theta T_b))\ \sin\ (\pi t / 2T_b)$$

$$= \pm\sqrt{2E_b / T_b}\sin\ (\pi t / 2T_b), \qquad 0 \le t \le 2T_b$$

+sign for $\theta\ (T_b) = \pi/2$ *and*
-sign for for $\theta\ (T_b) = -\pi/2$

1. The phase $\theta(0) = 0$ and $\theta(T_b) = \pi/2$, corresponds to transmission of symbol "1"
2. The phase $\theta(0) = \pi$ and $\theta(T_b) = \pi/2$, corresponds to transmission of symbol "0"
3. The phase $\theta(0) = \pi$ and $\theta(T_b) = -\pi/2$, (equivalently $3\pi/2$ modulo 2π) corresponds to transmission of symbol "1"
4. The phase $\theta(0) = 0$ and $\theta(T_b) = -\pi/2$, corresponds to transmission of symbol "0"

Thus MSK signal can take any one of four possible forms depending on the values of $\theta(0)$ and $\theta(T_b)$.

MSK are defined by a pair of sinusoidally modulated quadrature carriers as,

$$\varphi_1(t) = \sqrt{2/T_b}\ \cos(\pi t/2T_b)\cos(2\pi f_c t), \qquad 0 \le t \le T_b$$

$$\varphi_2(t) = \sqrt{2/T_b}\ \sin(\pi t/2T_b)\sin(2\pi f_c t), \qquad 0 \le t \le 2T_b$$

$$s(t) = s_1\,\varphi_1(t) + s_2(t)\,\varphi_2(t), \qquad 0 \le t \le T_b$$

The coefficients s_1 and s_2 are related to the phase states $\theta(0)$ and $\theta(T_b)$ respectively

$$s_1 = \int_{-T_b}^{T_b} s(t)\,\varphi_1(t)\,dt \quad = \sqrt{E_b}\cos(\theta(0)) \qquad -T_b \le t \le T_b$$

Both the integrals are evaluated over $2T_b$

$$s_2 = \int_{0}^{2T_b} s(t)\,\varphi_2(t)\,dt \quad = -\sqrt{E_b}\sin(\theta(T_b)) \quad 0 \le t \le 2T_b$$

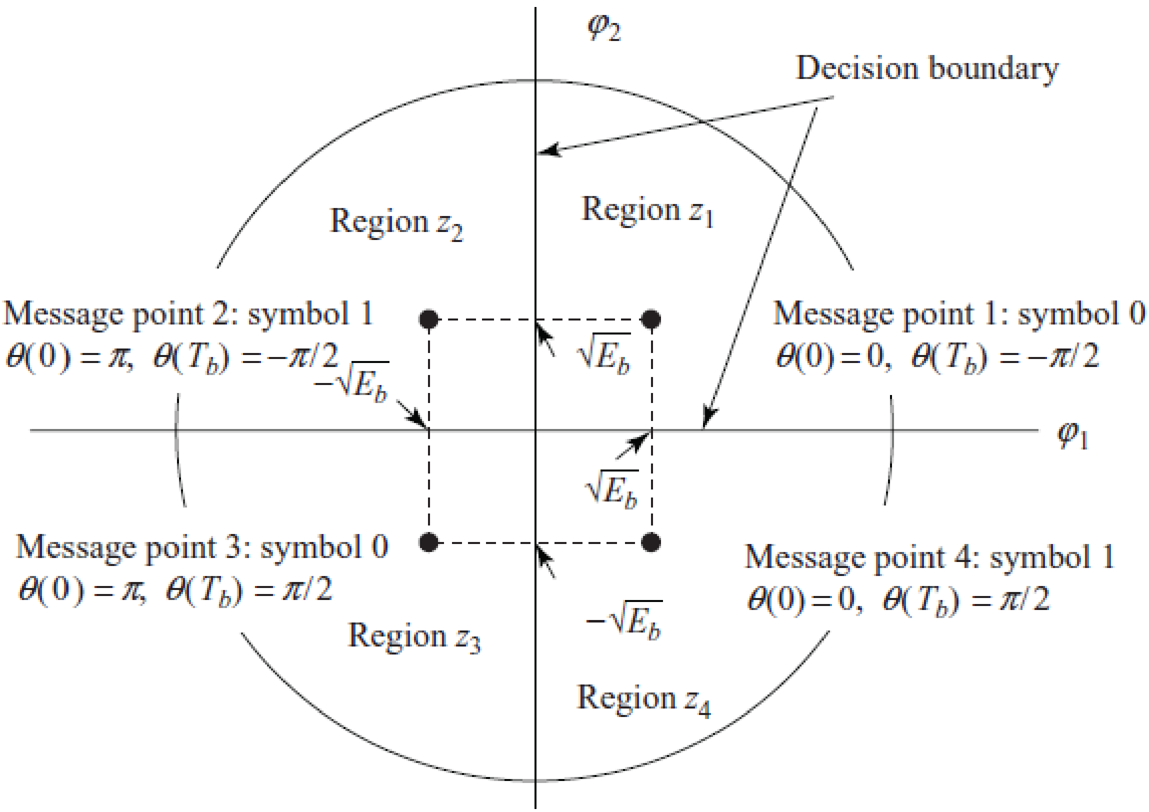
Signal Constellation of MSK waveforms

The signal constellation for an MSK signal is two-dimensional $(N = 2)$ with four possible message points $(M = 4)$. The coordinates of the four message points are:

$$(+\sqrt{E_b}, +\sqrt{E_b})\,(-\sqrt{E_b}, +\sqrt{E_b})\,(-\sqrt{E_b}, -\sqrt{E_b})\,(+\sqrt{E_b}, -\sqrt{E_b})$$

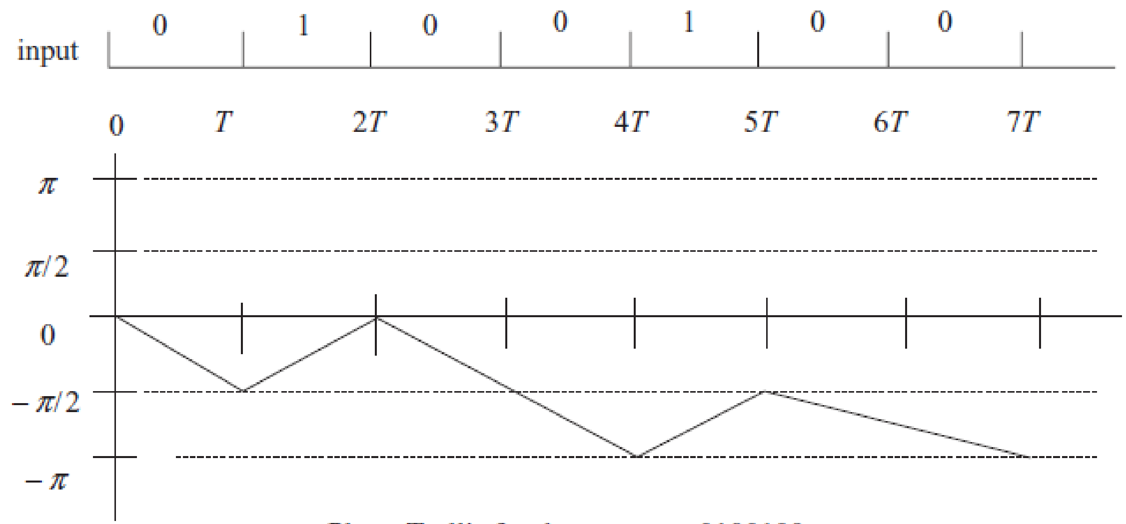
MSK signal-point diagram is similar to QPSK with a subtle difference.

In QPSK the transmitted symbol is represented by any one of the four message points, whereas in MSK one of two message points is used to represent the transmitted symbol at any one time, depending on the value of θ (0).

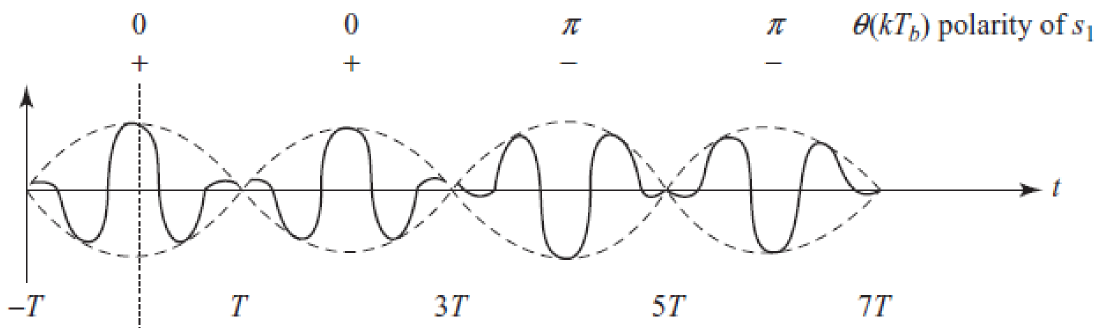


Signal space characteristics of MSK

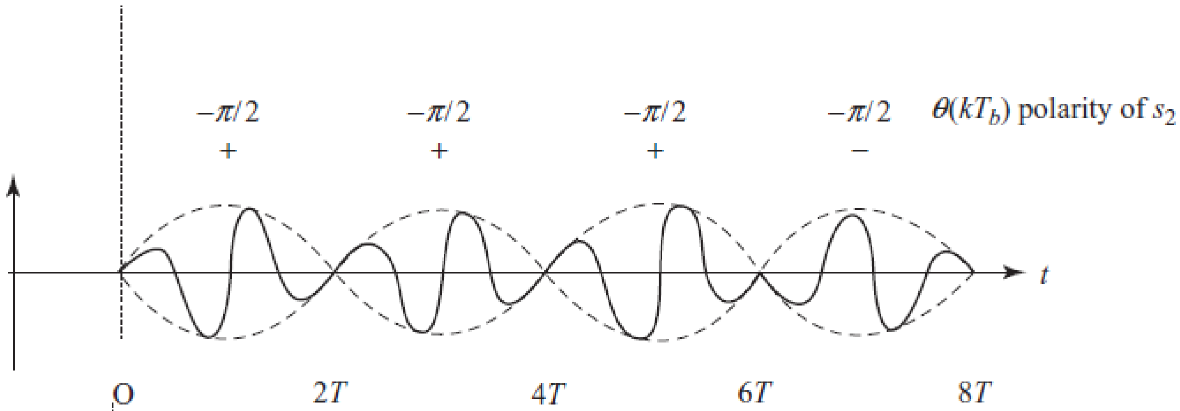
Transmitted binary symbol, $0 \le t \le T_b$	**Phase value in radians** $\theta(0)$	$\theta(T_b)$	**Message points coordinates** s_1	s_2
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$
1	π	$-\pi/2(3\pi/2)$	$-\sqrt{E_b}$	$+\sqrt{E_b}$
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$



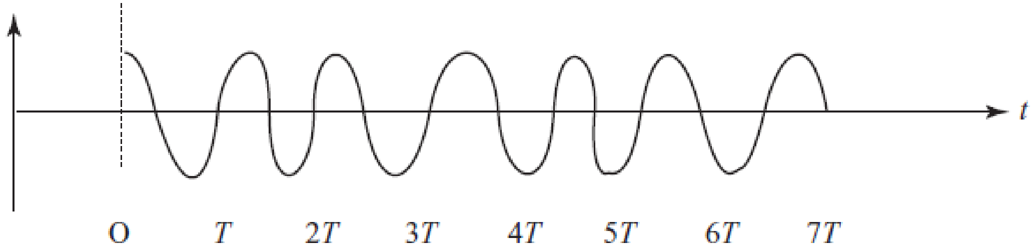
Phase Trellis for the sequence 0100100



(a) In-phase signal component



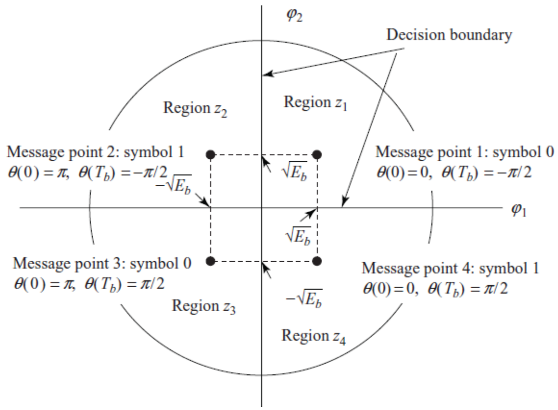
(b) Quadrature signal component



Error Probability of MSK Signal

In QPSK the transmitted signal is represented by any one of four messages, whereas in MSK one of two message points is used to represents transmitted signal at any time. The distance between two signal points in MSK is $\boxed{2\sqrt{E_b} \text{ and is same as of QPSK.}}$

receiver makes decision in between the message points m_1 and m_3 for symbol 1 and in between m_2 and m_4 for symbol 0 according to the estimation of $\theta(0)$ either 0 or π and $\theta(T_b)$ is $-\pi/2$ or $\pi/2$



Bit decisions are made alternately in the I (in-phase) and Q (quadrature) channels of the receiver over the period $2T_b$ seconds. The receiver makes an error whenever a wrong value of $\theta(0)$ in the I-channel or the wrong value of $\theta(T_b)$ in the Q-channel is obtained. The signal from other bits does not interfere with the receiver's decision for a given bit in a given channel. So, the bit error probability of MSK is exactly same as of QPSK or binary PSK signal and is given by

$$P_{e\,\textbf{symbol error}} = \operatorname{erfc}\left[\sqrt{E_b/N_0}\right]$$

$$P_{e\,\text{MSK}} = 1/2 \operatorname{erfc}\left[\sqrt{E_b/N_0}\right] \tag{5.120}$$

Generation and Detection of MSK Signals

At the input of MSK modulator there are two sinusoidal waves one with carrier frequency $f_c = n/4T_b$, *n is an integer, and other with frequency f = 1/4*T_b *those are applied at the product modulator.*

This produces two phase-coherent sinusoidal waves at frequencies f_1 *and* f_2 *related with* f_c *and* T_b *as*
$f_c = (f_1 + f_2)/2$, *and* $h = T_b\,(f_1 - f_2) = 0.5$. *Two narrowband filters are used to separate these frequencies.*

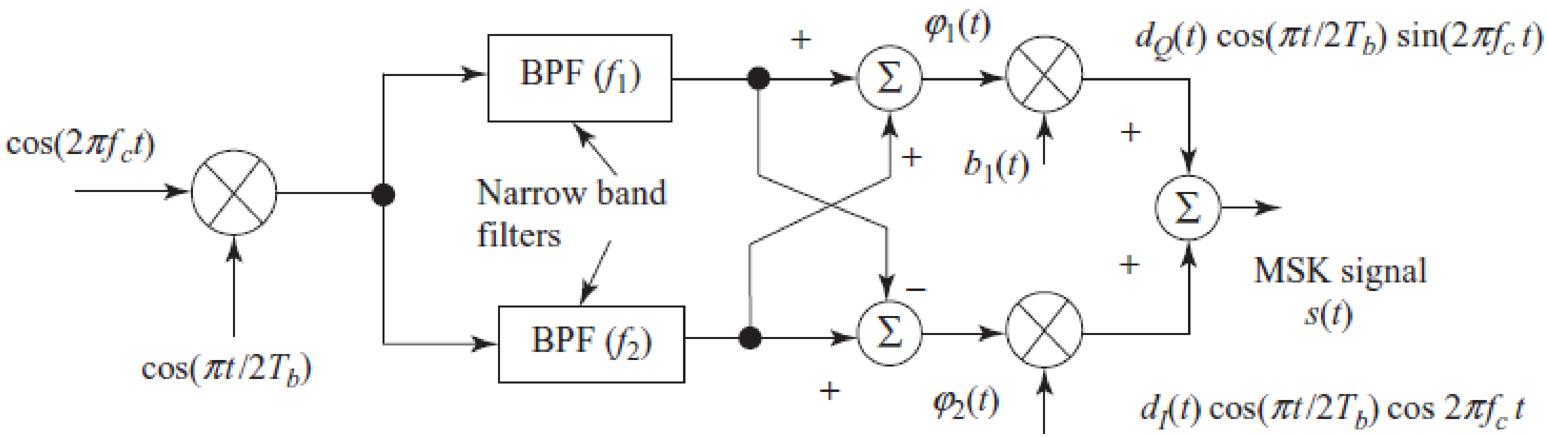
The resulting filter outputs are combined to produce two orthonormal signals $\boxed{\varphi_1(t) \text{ and } \varphi_2(t).}$

Multiplying $\varphi_1(t)$ and $\varphi_2(t)$ by

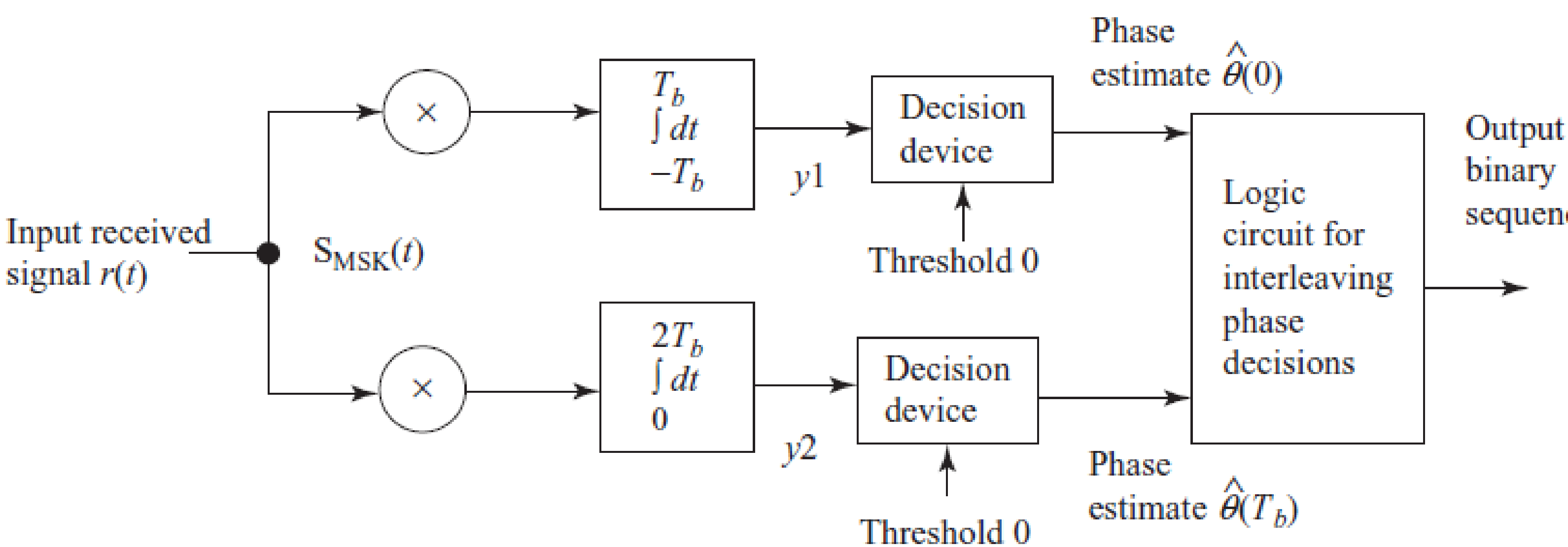
$\boxed{\text{two binary waves } b_1(t) \text{ and } b_2(t) \text{ with bit rate } 1/2T_b \text{ and added up to produce MSK signal.}}$

These binary waves

$\boxed{\text{are extracted from the information bit stream by using serial to parallel converter.}}$



(a) MSK modulator



Power spectra of MSK signals

The input binary wave is random in transmission of bits 1 and 0, i.e., equally likely. The symbols transmitted during different time slots being statistically independent. The inphase and quadrature components of the MSK signal is given by

$$s_I(t) = \sqrt{2\,E_b\,/T_b}\,\cos(\theta(t)) = \sqrt{2\,E_b\,/T_b}\,\cos(\theta(0))\cos(\pi t\,/2T_b)$$
$$= \pm\sqrt{2\,E_b\,/T_b}\,\cos(\pi t\,/2T_b), \qquad -T_b \le t \le T_b$$
$$s_Q(t) = \sqrt{2\,E_b\,/T_b}\,\sin(\theta(t)) = \sqrt{2\,E_b\,/T_b}\,\sin(\theta(T_b))\sin(\pi t\,/2T_b)$$
$$= \pm\sqrt{2\,E_b\,/T_b}\,\sin(\pi t\,/2T_b), \qquad 0 \le t \le 2T_b$$

Depending on the value of $\theta(0) = 0$, the in-phase component in terms of pulse shaping filter is + g (*t) and for $\theta(0) = \pi$ it is $-g$ (t) . In a similar way, we can find that the quadrature component s_Q (t) over the interval $0 \le t \le 2T_b$ is + g (t) or $-g$ (t) depending on phase state value $\theta(T_b)$ $+\pi/2$ and $-\pi/2$ respectively.*

$$g(t) = \begin{cases} \sqrt{2E_b\,/T_b}\,\cos(\pi t\,/2T_b), & -T_b \le t \le T_b \text{ for in-phase component} \\ 0, \text{ otherwise} & \end{cases}$$

$$g(t) = \begin{cases} \sqrt{2E_b\,/T_b}\,\sin(\pi t\,/2T_b), & 0 \le t \le 2T_b \text{ for quadrature component} \\ 0, \text{ otherwise} & \end{cases}$$

The energy spectral density for both of these components is equal to (obtained by Fourier transform),

$$\Psi_g(f) = (32\, E_b T_b)\, /\pi^2\, [\cos\, (2\pi f T_b) / (16\, T_b^{\,2} f^2 - 1)\,]^2 \qquad ($$

So, the power spectral density of the MSK signal is

$$S_{\mathrm{MSK}}(f) = 2 \times \Psi_g\ (\mathrm{f}) / 2\, T_b$$
$$= (32E_b\, /\pi^2)\, [\, \cos\, (2\pi f T_b) / (16\, T_b^{\,2} f^2 - 1)\,]^2$$

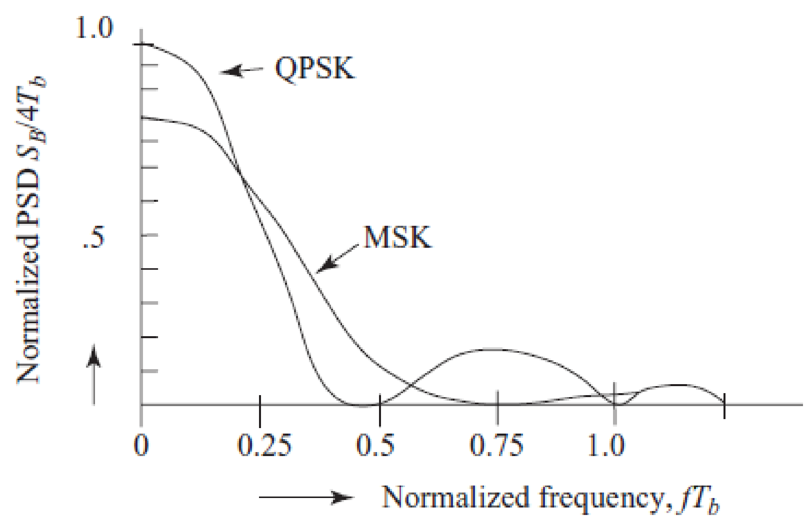
QPSK vs MSK

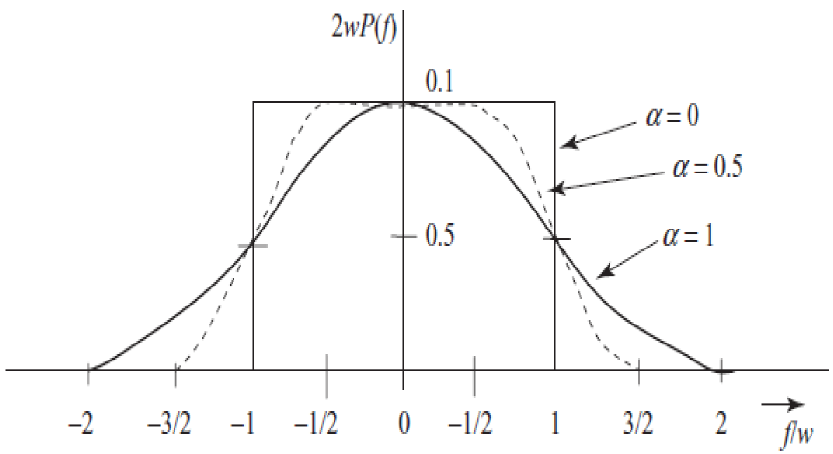
The MSK modulation makes the phase change linear and limited to ± (pi /2) over a bit interval T_b. *This enables MSK to provide a significant improvement over QPSK.*
MSK signal falls off as the inverse fourth power of frequency,
QPSK inverse quare of frequency
The main lobe of MSK becomes wider than the QPSK

But the fundamental problem with MSK is that the spectrum has side-lobes extending well above the data rate.

Wireless Channel requires more efficient use of RF channel Bandwidth, it is necessary to reduce the energy of the upper side-lobes.

As a solution to this problem a pre-modulation filter can be used (Low pass filter) or the more efficient and real approach is the use of Gaussian Filter that necessitates the use of GMSK in GSM wireless networks. Proper utilization of phase during detection is made in MSK for improving noise performance.





Orthogonal Frequency Division Multiplexing: OFDM
A choice for high rate transmission to reduce ISI effect

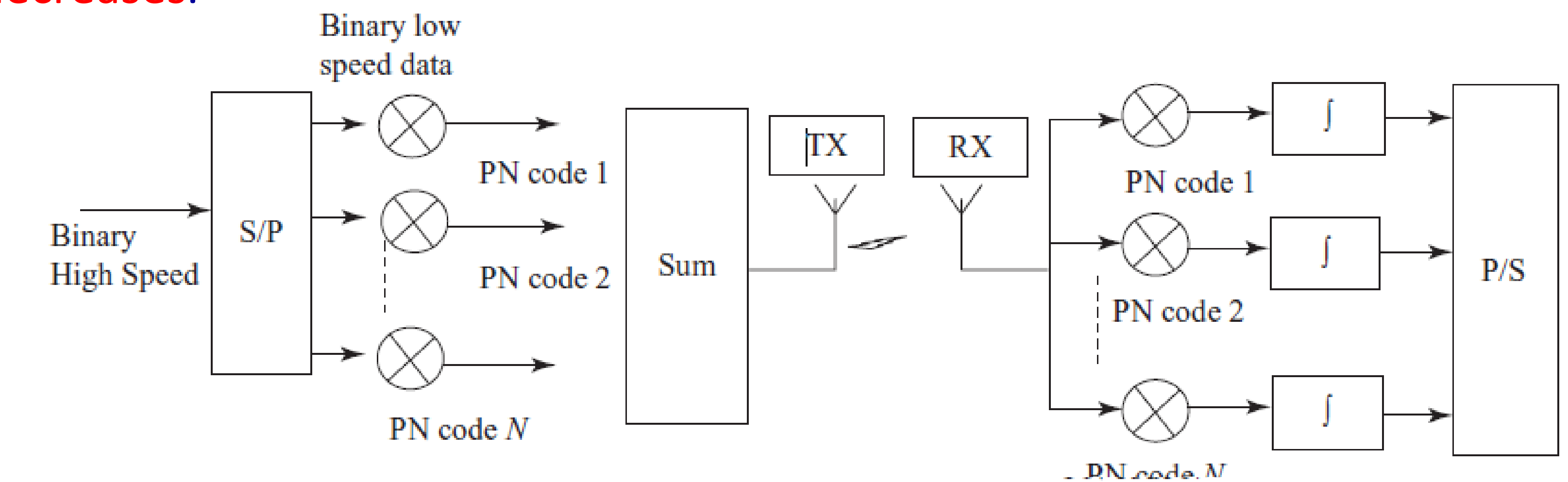
Single carrier vs. Multicarrier:
entire frequency bands is divided into number of subcarriers.

OFDM is the transmission technique used in code division multiple access (multicode), and (b) is the technique for multi-carriers where *N number of subcarriers are used.*

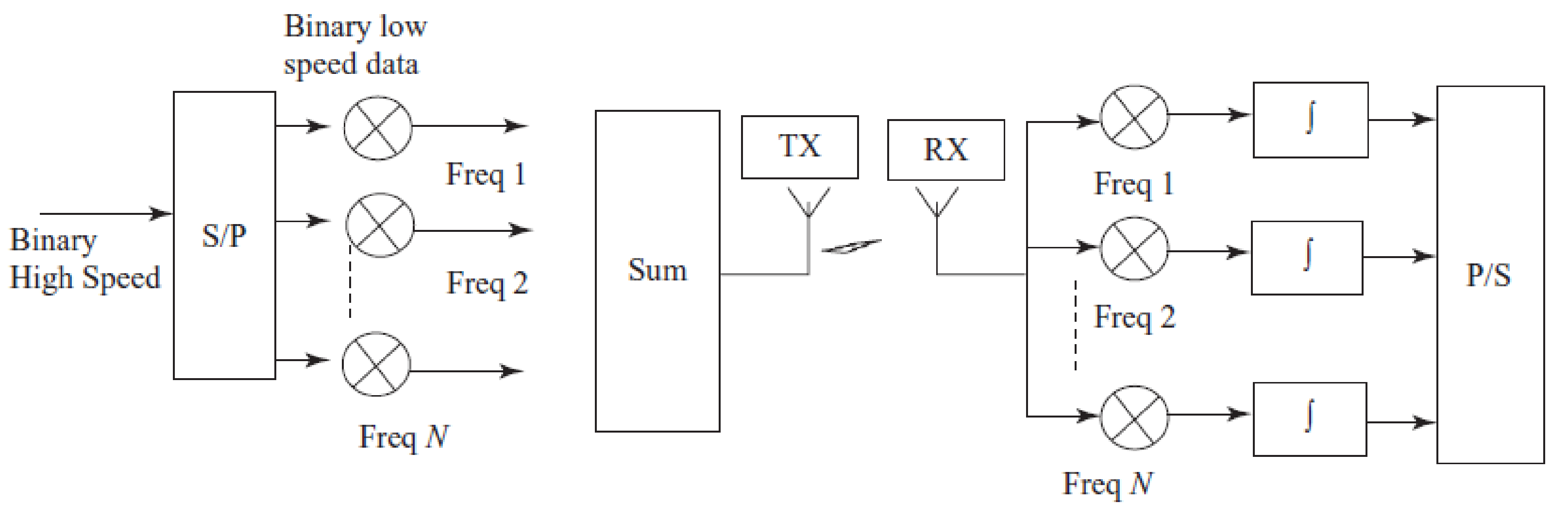
Now question arises why multicarrier than single carrier. The ISI is more in single carrier system as the channel is frequency selective, whereas ISI is greatly reduced in case of multicarrier transmission as flat fading occurs.

Higher delay spread leads to higher ISI. Multimedia transmission requires high data rate, so symbol duration is very small and signal is more and more affected by ISI when data rate increases.

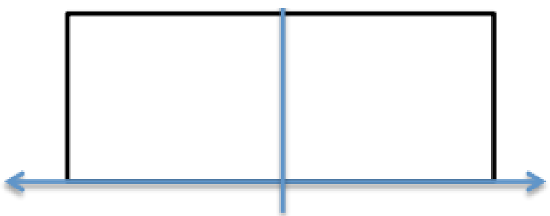
As multicarrier transmission divides the available carrier bandwidth into smaller sub-bands, represented by subcarriers whose data rate is smaller. So, symbol duration at the subcarrier increases and automatically ISI decreases.



(a)

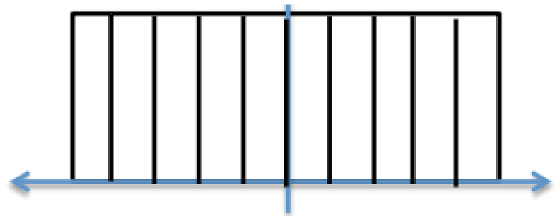


Wide-band channel



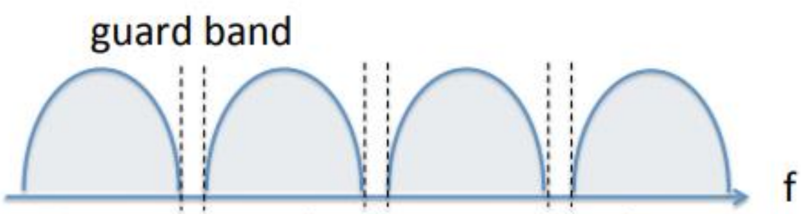
Send a sample using the entire band

Multiple narrow-band channels

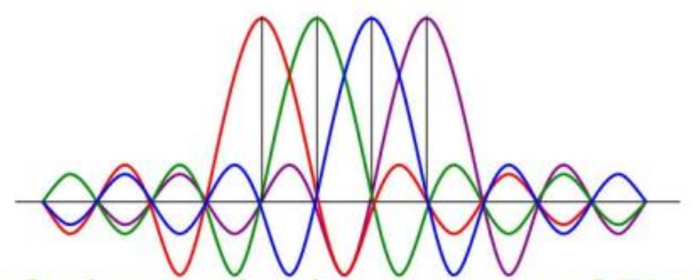


Send samples concurrently using multiple **orthogonal sub-channels**

- Multiple sub-channels (sub-carriers) carry samples sent at a lower rate
  - Almost same bandwidth with wide-band channel
- Only some of the sub-channels are affected by interferers or multi-path effect

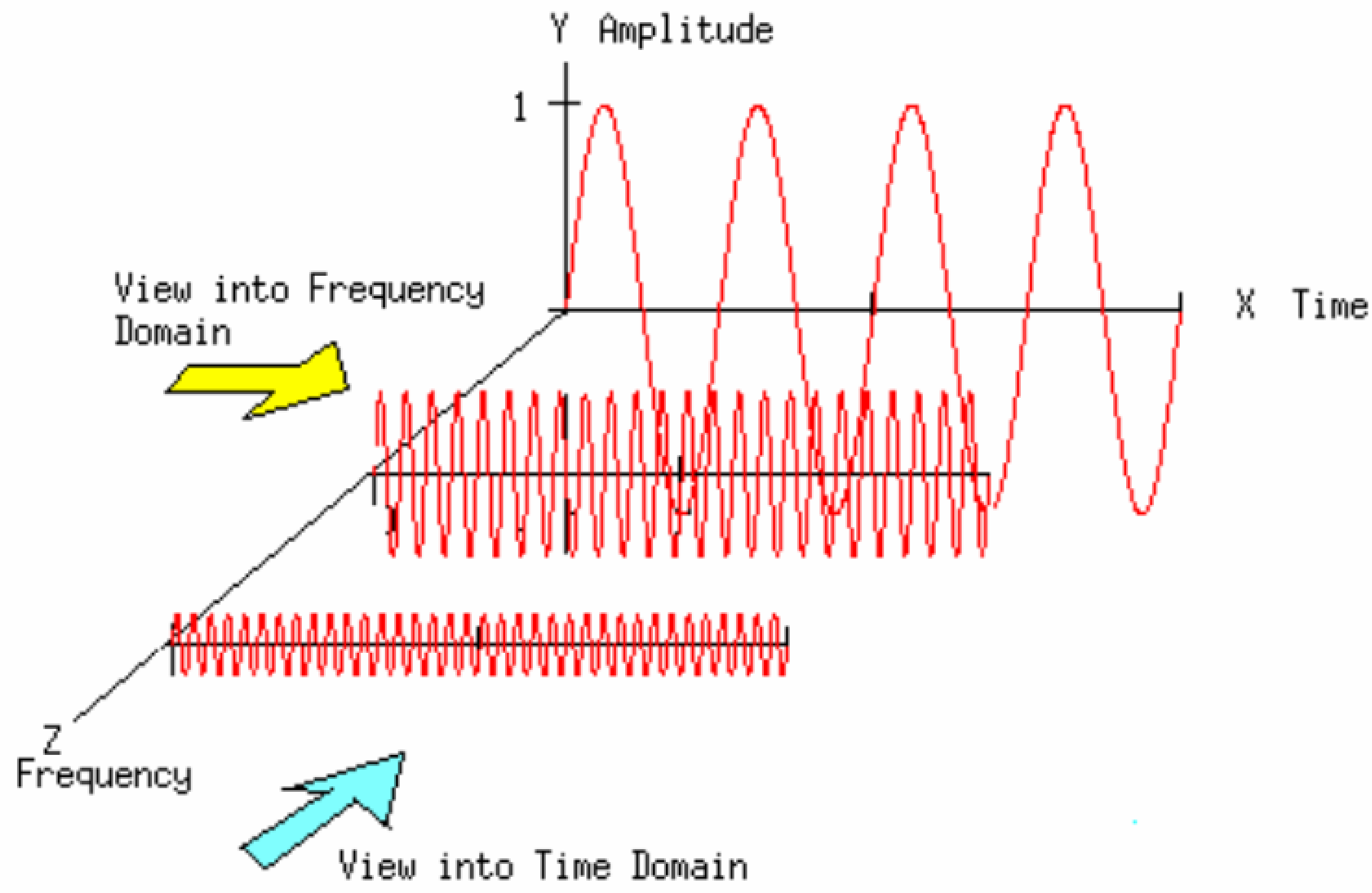


Frequency division multiplexing



Orthogonal sub-carriers in OFDM

Don't need guard bands


Y Amplitude
1
X Time
View into Frequency
Domain
Z
Frequency
View into Time Domain

Orthogonality of Sub-carriers

Encode: frequency-domain samples $\xrightarrow{\text{IFFT}}$ time-domain sample

$$x(t) = \sum_{k=-N/2}^{N/2-1} X[k] e^{j2\pi kt/N}$$

Time-domain

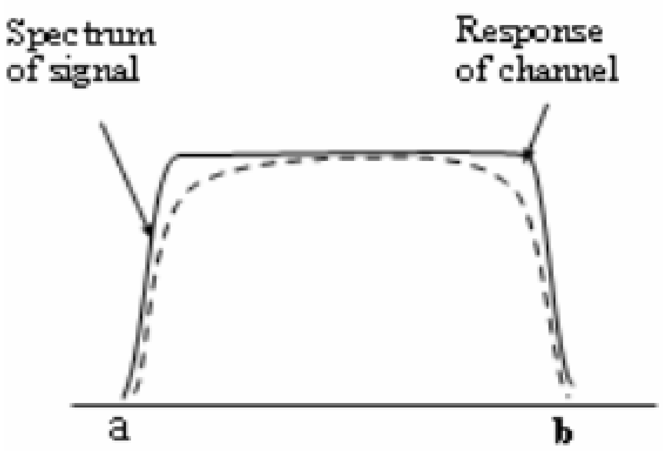
Frequency-domain

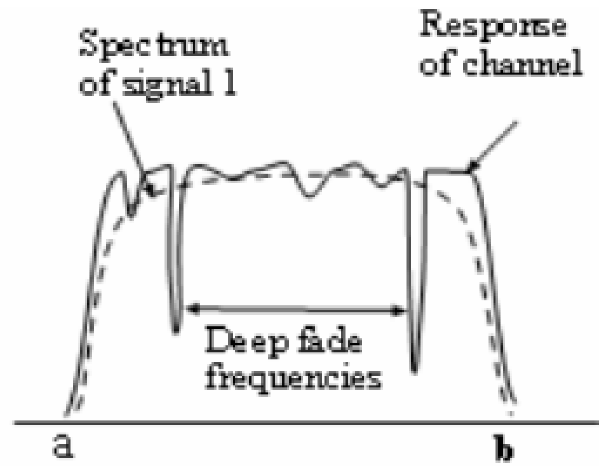
$$X[k] = \frac{1}{N} \sum_{t=N/2}^{N/2-1} x(t) e^{-j2\pi kt/N}$$

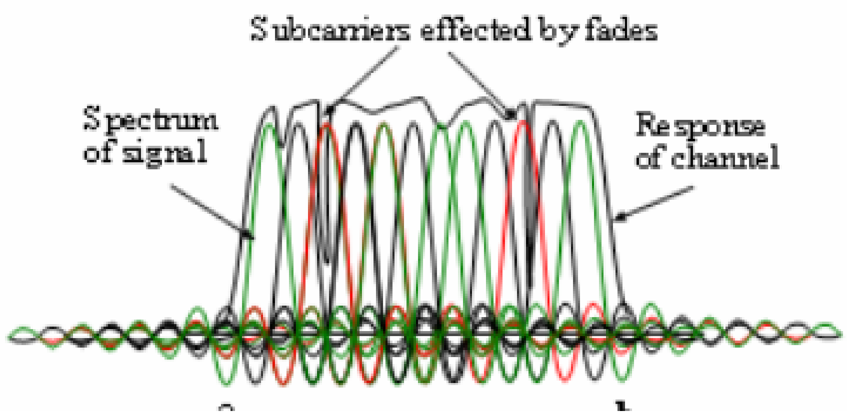
Decode: time-domain samples $\xrightarrow{\text{FFT}}$ frequency-domain sample

Orthogonality of any two bins : $\displaystyle\sum_{t=N/2}^{N/2-1} e^{-j2\pi kt/N} e^{-j2\pi pt/N} = 0, \forall p \neq k$

Frequency Selective Fading



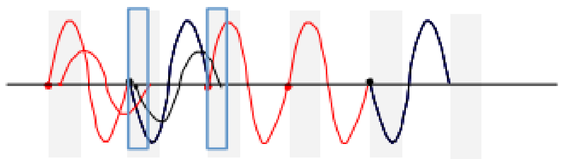




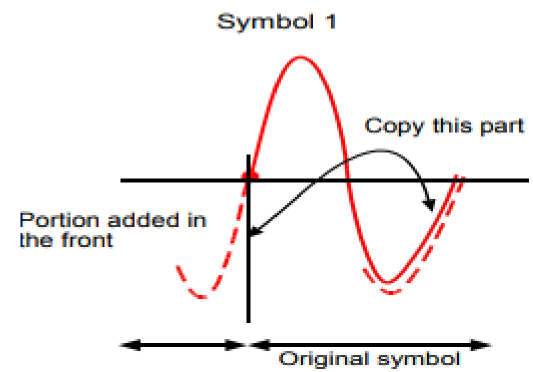
Frequency selective fading: Only some sub-carriers get affected

Inter Symbol Interference (ISI)

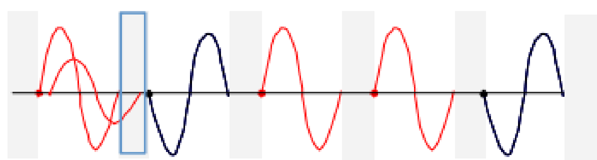
- The delayed version of a symbol overlaps with the adjacent symbol



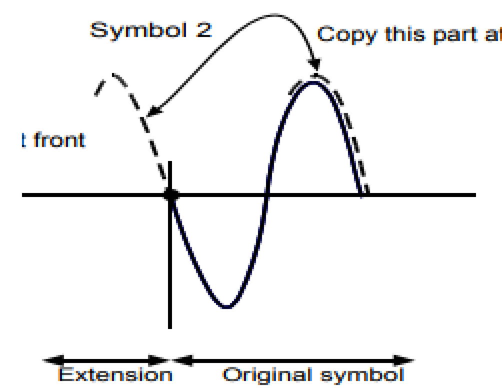
- Make the symbol period longer by copying the tail and glue it in the front



- One simple solution to avoid this is to introduce a guard-band



Guard band

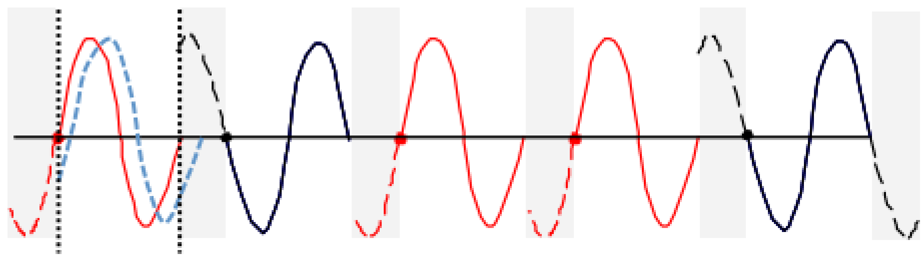


- However, we don't know the delay spread exactly

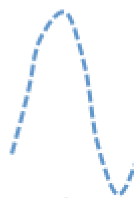
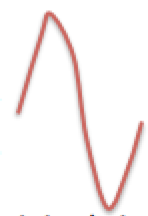
- The hardware doesn't allow blank space because it needs to send out signals continuously

Solution: Cyclic Prefix

Cyclic Prefix (CP)



- Because of the usage of FFT, the signal is periodic

FFT() = $\exp(-2j\pi_{\Delta}f)$*FFT()

delayed version

original signal

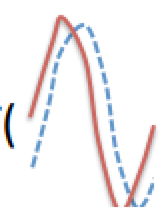
$Y[k] = \alpha(1+\exp(-2j\pi_{\Delta}k))*X[k]$
$= H'[k]X[k]$

- Delay in the time domain corresponds to rotation in the frequency domain
  - Can still obtain the correct signal in the frequency domain by compensating this rotation

Lump the phase shift in H

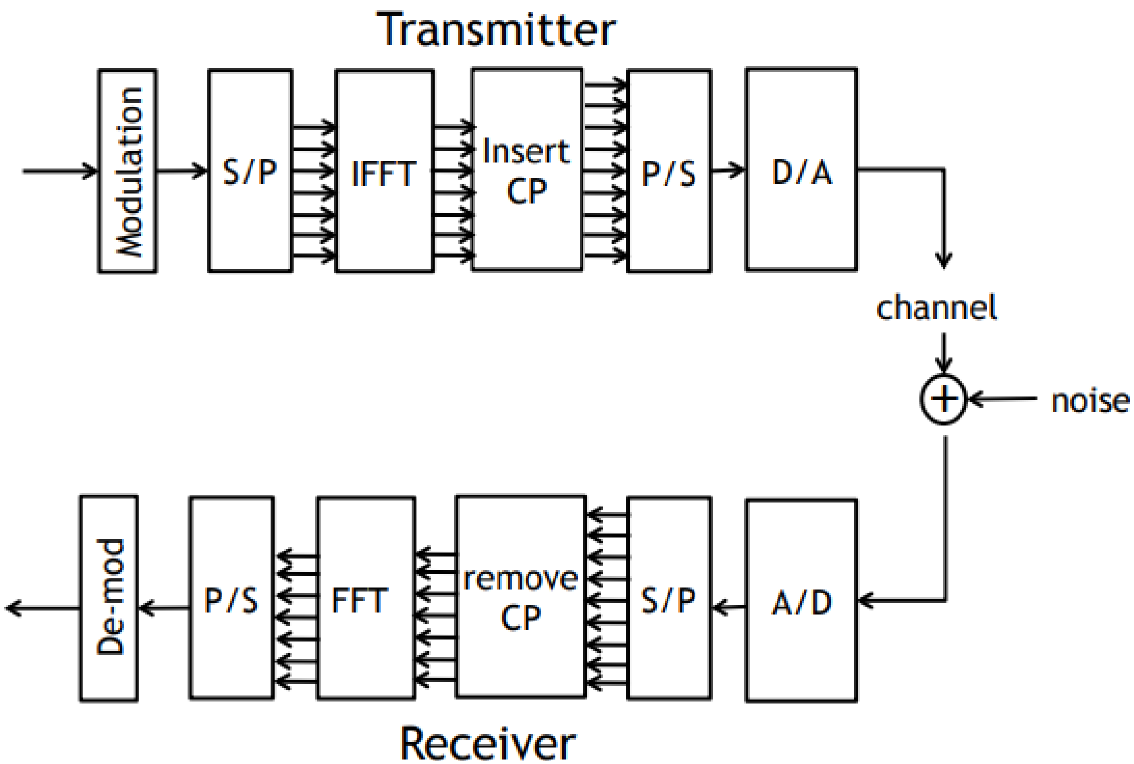
w/o multipath y(t) → FFT() → Y[k] = H[k]X[k]

original signal

w multipath y(t) → FFT()

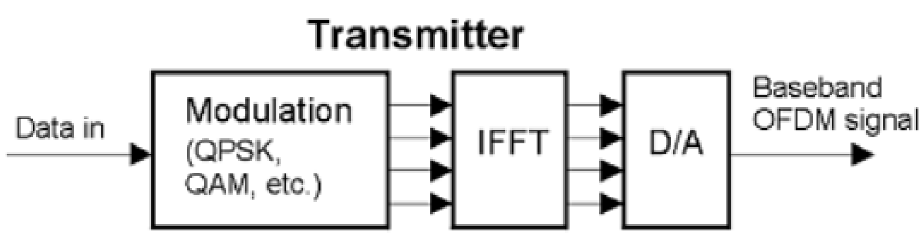
original signal + delayed-version signal

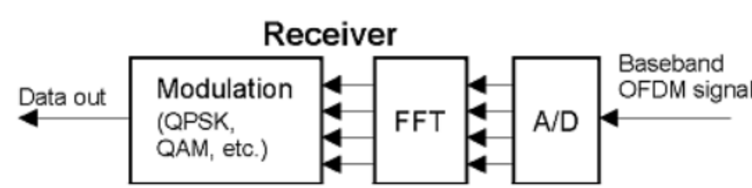
OFDM Diagram

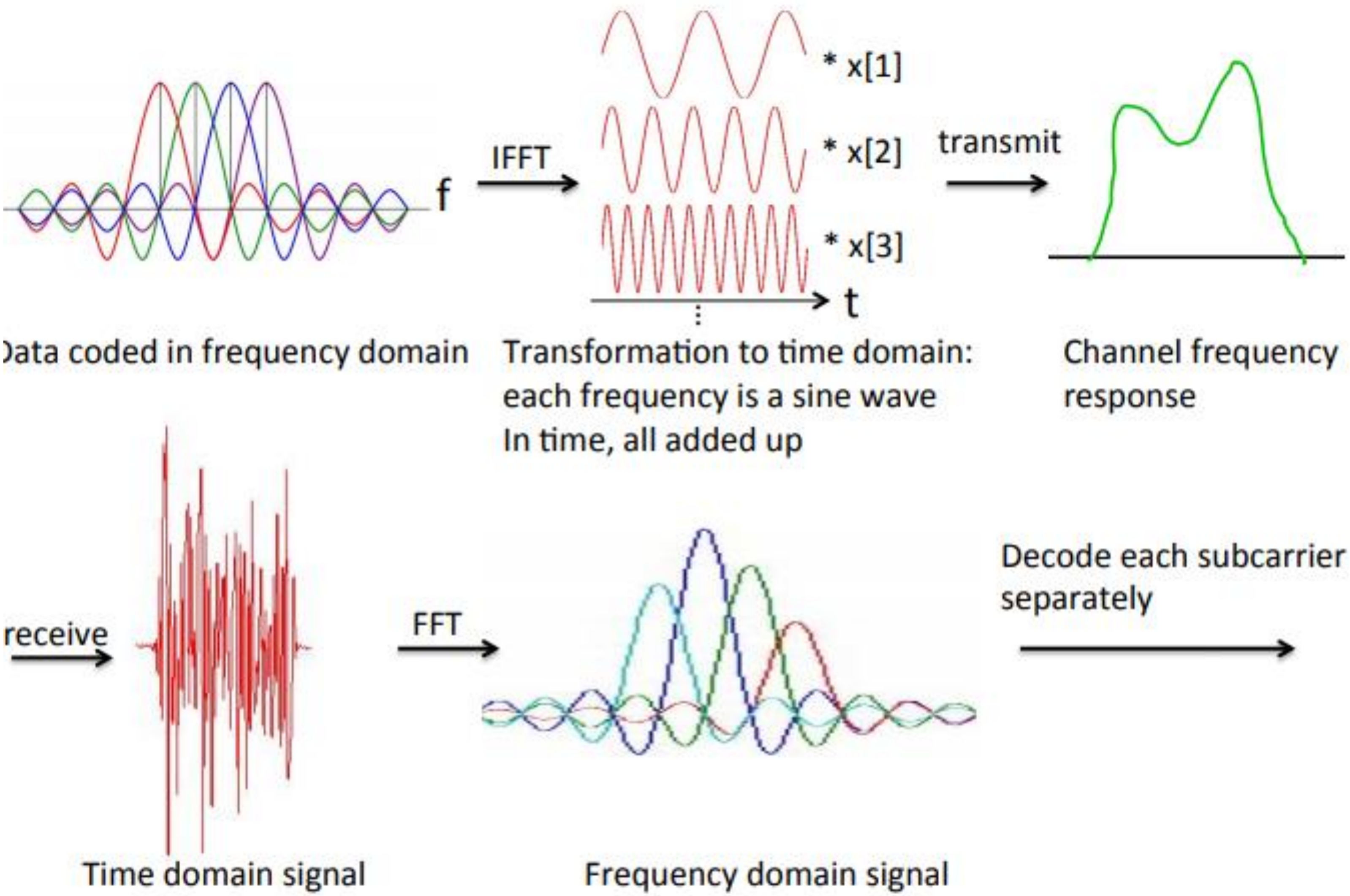


Applications for multimedia services

All physical layer for WLAN, WiMAX, LTE
LTE Advanced use OFDM multiplexing

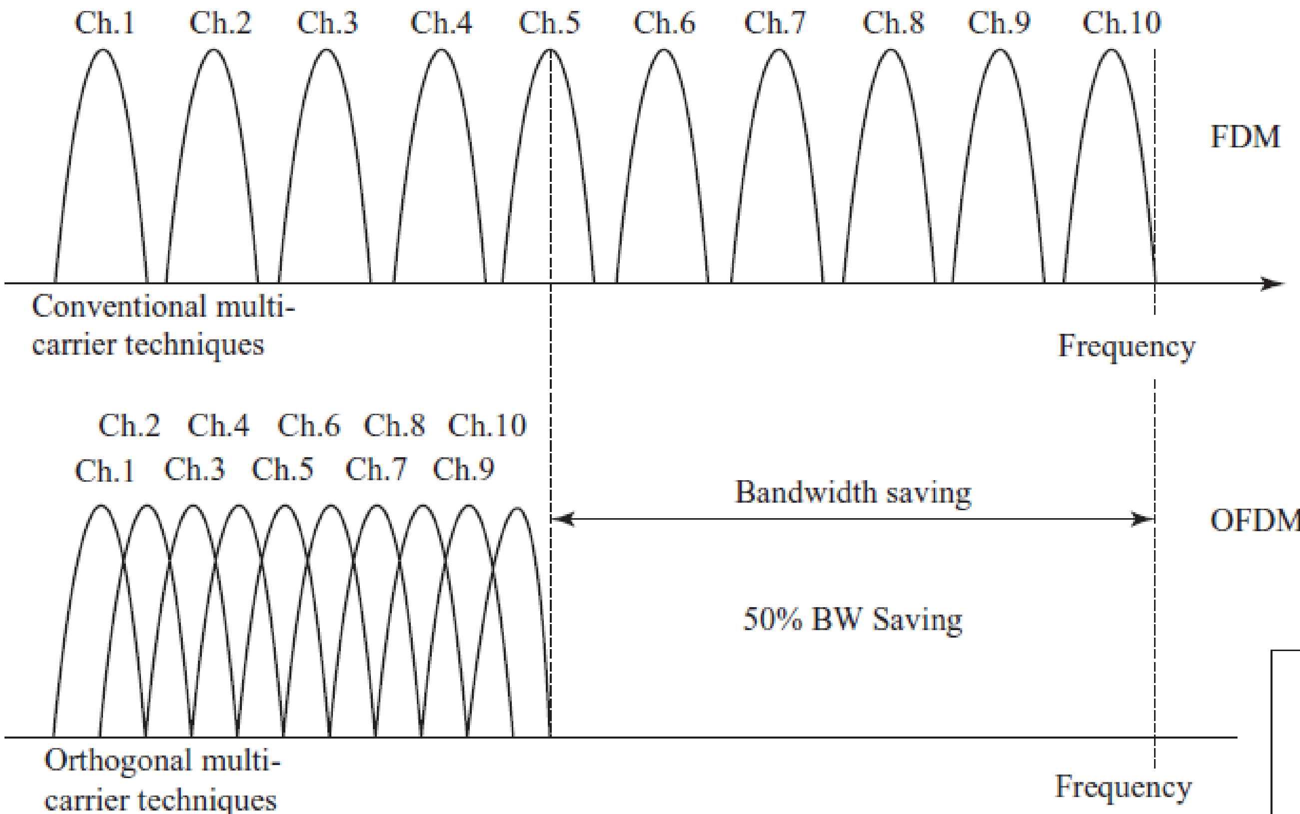


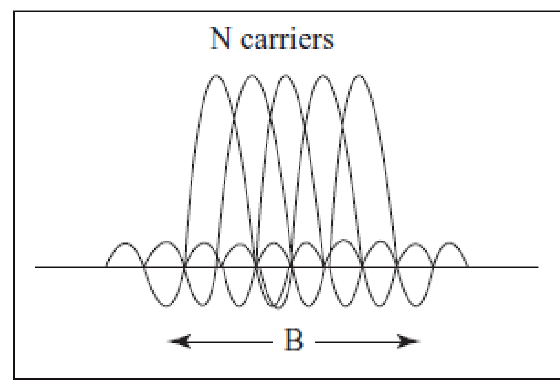




f
IFFT
* x[1]
* x[2]
* x[3]
t
transmit
Data coded in frequency domain
Transformation to time domain:
each frequency is a sine wave
In time, all added up
Channel frequency
response
receive
FFT
Decode each subcarrier
separately
Time domain signal
Frequency domain signal

OFDM – Orthogonal FDM, saves 50 band width
Higher data rate is supported





2. An NRZ data stream (of amplitude ± 1) is passed through a LPF whose impulse response is defined by the Gaussian function $h(t) = \frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\pi^2 t^2}{\alpha^2}\right)$

$$\alpha = \sqrt{\frac{\log 2}{2}} \cdot \frac{1}{\omega}$$ design parameter in terms of the filter's 3-dB bandwidth

Show that $(f) = \exp(-\alpha^2 f^2)$. Find the 3-dB bandwidth of the filter. Explain the significance.

Solution given

$$h(t) = \frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\pi^2 t^2}{\alpha^2}\right)$$

From Fourier transform tables,

$$\left[\exp(-k t^2)\right] = \sqrt{\frac{\pi}{k}} \exp\left(-\frac{\pi^2 f^2}{k}\right)$$

replace k with $\frac{\pi^2}{\alpha^2}$

$$\mathcal{F}\left[h(t)\right] = H(f) = \frac{\sqrt{\pi}}{\alpha} \cdot \sqrt{\frac{\pi}{\pi^2/\alpha^2}} \exp\left(\frac{-\pi^2 f^2}{\frac{\pi^2}{\alpha^2}}\right)$$

$$= \frac{\cancel{\sqrt{\pi}}}{\cancel{\alpha}} \cdot \frac{\cancel{\alpha}}{\cancel{\sqrt{\pi}}} \cdot \exp\left(-\alpha^2 f^2\right) = \exp\left(-\alpha^2 f^2\right), \text{ as required.}$$

For finding the 3-dB BW,

$$\left(\frac{\cancel{\log 2}}{2}\right) \cdot \frac{1}{\omega^2} \cdot f^2 = \cancel{\left(\frac{\log 2}{2}\right)} \Rightarrow \omega = f \quad \text{or } f = \omega$$

$$\exp\left(-\alpha^2 f^2\right) = \frac{1}{\sqrt{2}} \Rightarrow \exp\left(+\alpha^2 f^2\right) = \sqrt{2}$$

$$\alpha^2 f^2 = \log\sqrt{2} = \frac{\log 2}{2}$$

$$\alpha = \sqrt{\frac{\log 2}{2}} \cdot \frac{1}{\omega}$$

$$\left(\frac{\cancel{\log 2}}{2}\right) \cdot \frac{1}{\omega^2} \cdot f^2 = \cancel{\left(\frac{\log 2}{2}\right)} \Rightarrow \omega = f \quad \text{or } f = \omega$$

3-dB BW of the filter ω

3. The signal vectors $\vec{s}_1$ and $\vec{s}_2$ are used to represent binary symbols 1 and 0, respectively, in a coherent binary FSK system. The receiver decides in favor of symbol 1 when $\vec{x}^T\vec{s}_1 > \vec{x}^T\vec{s}_2$, where $\vec{x}^T\vec{s}_i$ is the inner product of the observation vector $\vec{x}$ and the signal vector $\vec{s}_i$, where $i = 1, 2$. Show that this decision rule is equivalent to the condition $x_1 > x_2$, where x_1 and x_2 are the elements of the observation vector $\vec{x}$. Assume that the signal vectors $\vec{s}_1$ & $\vec{s}_2$ have equal energy.

SOLUTION :— 3

The 2-message points are defined by the signal

$$\vec{s}_1 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}$$

where E_b is the transmitted signal energy per bit.

The observation vector $\vec{x}$ is given by $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \quad \vec{x}^T \vec{S_1} > \vec{x}^T \vec{S_2} \quad \Rightarrow \quad x_1\sqrt{E_b} > x_2\sqrt{E_b} \quad \Rightarrow x_1 > x_2 .$$

Hence, the condition $\vec{x}^T \vec{S_1} > \vec{x}^T \vec{S_2}$ is equivalent to $x_1 > x_2$ provided $\vec{S_1}$ and $\vec{S_2}$ have equal energy.

Problem 3

In a coherent FSK system, $s_1(t)$ and $s_2(t)$ representing symbols 1 and 0

$$s_1(t),\ s_2(t) = A_c \cos\left[2\pi\left(f_c \pm \frac{\Delta f}{2}\right)t\right] \quad ; \ 0 \leq t \leq T_b$$

Assuming that $f_c > \Delta f$

show that the correlation coefficient of the signals $s_1(t)$ & $s_2(t)$ is approximated

$$\rho = \frac{\int_0^{T_b} s_1(t)\, s_2(t)\, dt}{\int_0^{T_b} s_1^2(t)\, dt} \simeq \text{sinc}\,(2\,\Delta f\, T_b)$$

a. Find minimum frequency shift Δf for which s1 and s2 are orthogonal.
b. The value of Δf that minimizes the average bit error probability.
c. Find the value of Eb/No for this Δf so that the error probability for coherent FSK system has some noise performance as a coherent binary PSK system.

$$s_1(t) = A_c \cos\left[2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right] \quad \text{for symbol } 1\ ;\ 0 \leq t \leq T_b.$$

$$s_2(t) = A_c \cos\left[2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right] \quad \text{for symbol } 0\ ;\ 0 \leq t \leq T_b.$$

$$\rho = \frac{\int_0^{T_b} s_1(t)\, s_2(t)\, dt}{\int_0^{T_b} s_1^2(t)\, dt} = \frac{\int_0^{T_b} \left[\cancel{A_c} \cos\left[2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right]\right]\left[\cancel{A_c} \cos\left[2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right]\right] dt}{\int_0^{T_b} \cancel{A_c^2} \cos^2\left[2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right] dt}$$

$$= \frac{\int_0^{T_b} \left[\cos\left(4\pi f_c t\right) + \cos\left(2\pi \Delta f t\right)\right] dt}{\int_0^{T_b} 2\cos^2\left[2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right] dt}$$

$$= \frac{\int_0^{T_b} \left[\cos\left(4\pi f_c t\right) + \cos\left(2\pi \Delta f t\right)\right] dt}{\int_0^{T_b} \left(1 + \cos\left[4\pi\left(f_c + \frac{\Delta f}{2}\right)t\right]\right) dt}$$

$$= \frac{\dfrac{\sin 4\pi f_c T_b}{4\pi f_c} + \dfrac{\sin 2\pi \Delta f T_b}{2\pi \Delta f}}{T_b + \dfrac{\sin\left\{4\pi\left(f_c + \frac{\Delta f}{2}\right)T_b\right\}}{4\pi\left(f_c + \frac{\Delta f}{2}\right)}}$$

Since $f_c >> \Delta f$

neglect the first term in the Numerator

the second term in the denominator

$$\rho \simeq \frac{\sin(2\pi \Delta f T_b)}{2\pi \Delta f T_b} = \text{sinc}(2\Delta f T_b) \quad \left(\because \frac{\sin(\pi x)}{\pi x} = \text{sinc}(x)\right)$$

For orthogonality, $\rho = 0$ $\text{sinc}(2\Delta f T_b) = 0$ $\Rightarrow \dfrac{\sin(2\pi \Delta f T_b)}{2\pi \Delta f T_b} = 0.$

$$\Rightarrow 2\cancel{\pi} \Delta f T_b = n\cancel{\pi} \quad \Rightarrow \quad \Delta f = \frac{n}{2T_b}. \quad (n = 0, 1, 2, \ldots.)$$

Therefore, we find that for $s_1(t)$ & $s_2(t)$ to be orthogonal, $\Delta f|_{min.} = \dfrac{1}{2T_b}$

minimum value of $\Delta f = \dfrac{1}{2T_b}. = \Delta f|_{min.}$

for a coherent FSK with correlation coefficient ρ

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right)$$

Hence, for minimum P_e, $\mathrm{erfc}\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right)$ has to be minimum.

$(1-\rho)$ has to be maximum

($\because$ erfc is a decreasing function) $\Rightarrow \rho$ has to be minimum.

Hence, $\rho = \mathrm{sinc}\,(2\Delta f T_b) = \dfrac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}$.

Say $2\Delta f T_b = x$ Then $\rho = \mathrm{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$

$$\Rightarrow \frac{d\rho}{dx} = \frac{\pi x . \pi\cos\pi x - \pi\sin\pi x}{\pi^2 x^2}$$

for minimum, $\dfrac{d\rho}{dx} = 0$

$$\Rightarrow \sin \pi x = \pi x \cos \pi x \Rightarrow \tan \pi x = \pi x \longrightarrow (1)$$

This equation can be solved by numerical methods, and the minimum positive value of x for which (1) is satisfied is $x = 1.43$.

Also, when $x = 1.43$, $\rho = \rho|_{min} = -0.217$.

$$\Rightarrow x = 1.43 \Rightarrow 2\Delta f T_b = 1.43 \Rightarrow \Delta f = \frac{1.43}{2T_b} = \frac{0.715}{T_b}$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b(1+0.217)}{2N_0}}\right)$$

$$= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{0.6085 E_b}{N_0}}\right)$$

BPSK system, $P_e|_{BPSK} = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

Hence, for this particular value of Δf, for $P_e|_{BPSK} = P_e$

$\frac{E_b}{N_0}$ has to increased $\frac{1}{0.6085} = 1.643$ times. $= 10\log_{10}(1.643)$
$\cong 2.16\ dB$

5. The PSK signal $s(t) = A_c k \sin(2\pi f_c t) \pm A_c\sqrt{1-k^2}\cos(2\pi f_c t)$ over $0 \leq t \leq T_b$ where '+' sign corresponds to symbol 1 and '−' sign corresponds to symbol 0. The first term represents a carrier component included for the purpose of synchronization of the receiver to the transmitter

Draw the signal space diagram of the system and explain observation

zero mean AWGN of PSD $\frac{N_0}{2}$

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b(1-k^2)}{N_0}}\right)$$

where $E_b = \frac{1}{2}A_c^2 T_b$

If 10 % of the transmitted signal is allocated to the carrier component, determine the required Eb/No for the error probability to remain 10^{-4}

Compare the value of Eb/No with that required for a conventional PSK system with the same probability of error.

$$s_1(t) = A_c\sqrt{1-k^2}\cos(2\pi f_c t) + A_c k \sin(2\pi f_c t). \text{ for symbol 1.}$$

$$s_2(t) = -A_c\sqrt{1-k^2}\cos(2\pi f_c t) + A_c k \sin(2\pi f_c t). \text{ for symbol 0.}$$

In terms of E_b and T_b, we may write :-

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\sqrt{1-k^2}\cos(2\pi f_c t) + \sqrt{\frac{2E_b}{T_b}}\, k \sin(2\pi f_c t) \text{ .... for 1.}$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}}\sqrt{1-k^2}\cos(2\pi f_c t) + \sqrt{\frac{2E_b}{T_b}}\, k \sin(2\pi f_c t) \text{ .... for 0.}$$

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \qquad \phi_2(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t),$$

$$s_1(t) = \sqrt{E_b(1-k^2)}\ \phi_1(t) + \sqrt{k^2 E_b}\ \phi_2(t).$$

$$s_2(t) = -\sqrt{E_b(1-k^2)}\ \phi_1(t) + \sqrt{k^2 E_b}\ \phi_2(t)$$

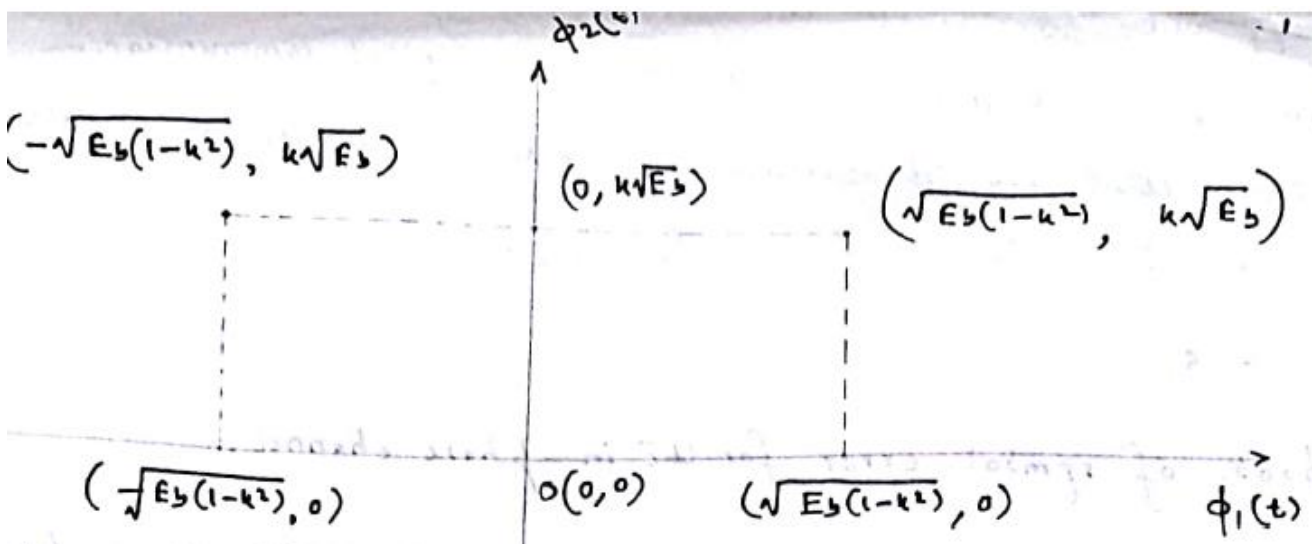
$$\left.\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t) \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t) \end{aligned}\right\}$$

$$s_{11} = \sqrt{E_b(1-k^2)}$$

$$s_{21} = -\sqrt{E_b(1-k^2)}$$

the signal-space diagram

$$s_{12} = s_{22} = k\sqrt{E_b}$$



introduction of the carrier component for synchronization makes the signal space diagram two-dimensional

if k is zero, then the problem becomes one-dimensional BPSK scheme.

binary PSK

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d_{in}}{2\sqrt{N_0}}\right)$$

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\cancel{2}\sqrt{E_b(1-k^2)}}{\cancel{2}\sqrt{N_0}}\right)$$

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b(1-k^2)}{N_0}}\right)$$

If 10% of the transmitted signal power is allocated to the carrier, since power $\propto$ (amplitude)^2, $k^2 = 10\% = 0.1$.

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b(1-k^2)}{N_0}}\right) \Rightarrow 10^{-4} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{0.9E_b}{N_0}}\right)$$

$$\mathrm{erfc}\left(\sqrt{\frac{E_b \times 0.9}{N_0}}\right) = 2\times 10^{-4} \qquad \sqrt{\frac{0.9E_b}{N_0}} = \mathrm{erfc}^{-1}(2\times 10^{-4}) = 2.63$$

$$\Rightarrow \frac{E_b}{N_0} = \frac{2.63^2}{0.9} = 7.685 \cong 8.86\,dB$$

For conventional PSK, $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \Rightarrow \mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = 2\times 10^{-4}$

$$\frac{E_b}{N_0} = 2.63^2 = 6.917 \simeq 8.4\,dB$$

BPSK system requires 0.46 dB less $\frac{E_b}{N_0}$ for the same average probability of symbol error.