Digital Signal Processing

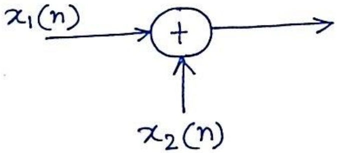
M. K. Naskar

Different representations

- Block diagram
- System functional
- Difference equation
- Impulse response

Elements of a Digital Filter

- Adder :

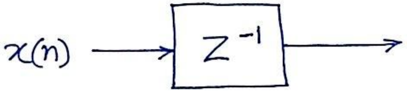


$y(n) = x_1(n) + x_2(n)$

- Multiplier :

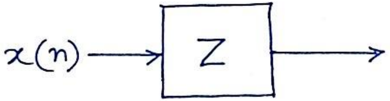
$y(n) = a \cdot x(n)$

- Delay :



$y(n) = x(n-1)$

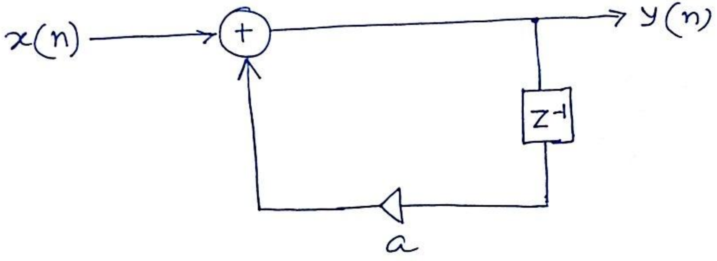
- Advance :



$y(n) = x(n+1)$

Characterization of Digital Filters

(Difference eqn. representation of a digital filter)
(Process of analysis)



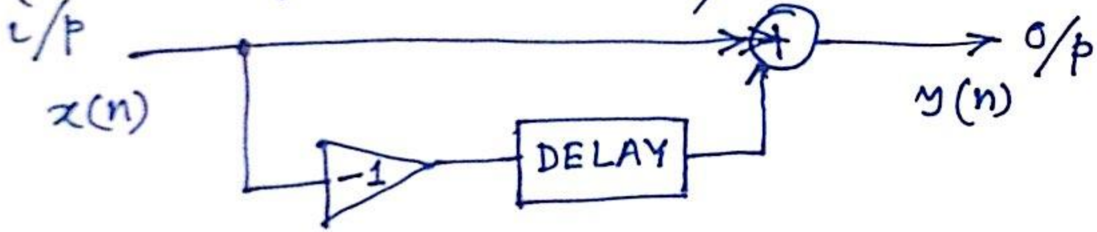
Difference equation ⇒

$$y(n) = x(n) + a\,y(n-1)$$

~~So, $y(n) = G[x(n)] = x(n) + a\,y(n-1)$~~

Prob. Block diagram of a DT system is given below. Sketch its unit-sample response.

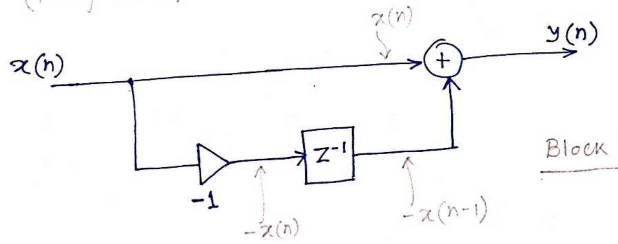
Assume that system is initially at rest i.e. o/p of the DELAY is 0;



- Non-recursive (Feedforward)
- 1^{st} order

First-order Non-recursive Filter

(Feedforward)



Block Diagram

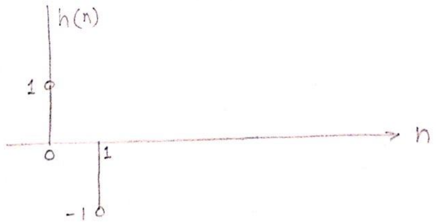
Formal Approach :

$$y(n) = x(n) - x(n-1)$$ ← Difference equation

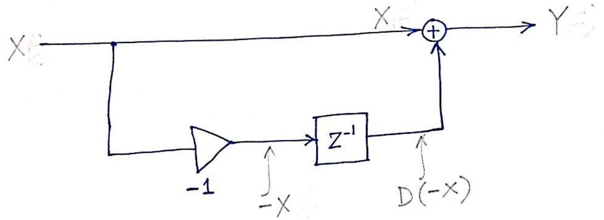
$$\Rightarrow \quad h(n) = \delta(n) - \delta(n-1)$$

Impulse (unit sample) response.

Sketch of h(n) :



First-order Non-recursive Filter



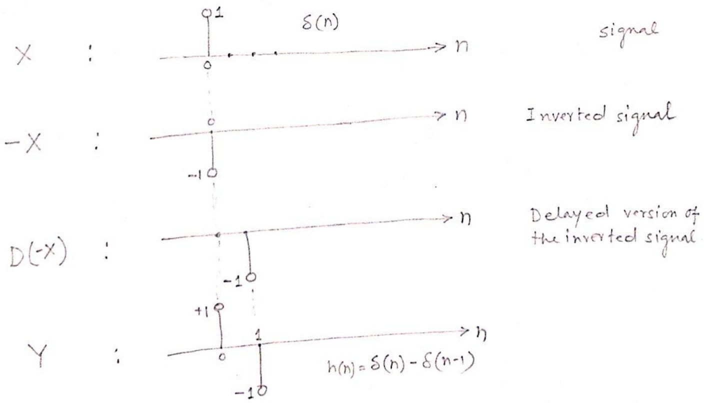
Operator approach : (signal as a whole)

Delay operator

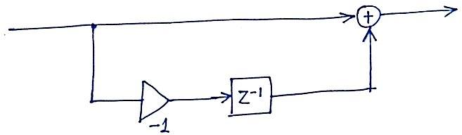
$$Y = X + D(-X)$$

$$\Rightarrow Y = X - DX = (1-D)X$$

This gives the algorithm to obtain Y from X.

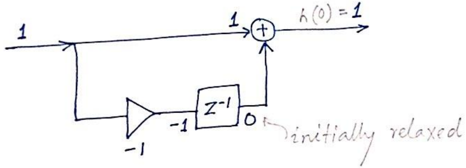


First-order Non-recursive Filter

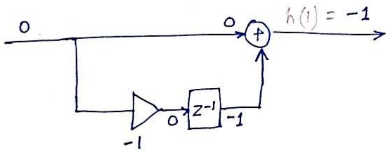


Informal Approach (Sample by sample analysis)
(Intuitive)

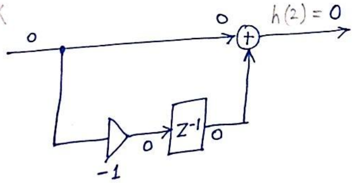
At $n=0$,



After clocking
↓
At $n=1$,

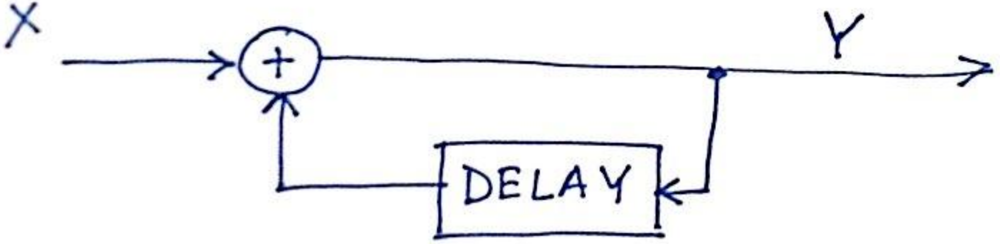


After another clock
↓
At $n=2$



Hence, $h(n) = \{\underset{\uparrow}{1}, -1, 0, \cdots\} = \delta(n) - \delta(n-1)$

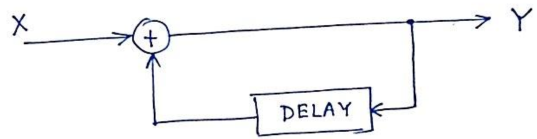
Prob. Block diagram of DT system is given below. Sketch the i/p sequence for which the o/p is a unit step sequence. Assume that the system is initially at rest.



· Recursive
· 1st order

First-order Recursive Filter:

(Feedback)



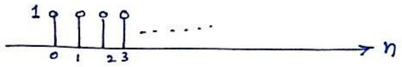
First-order Filter

$Y = X + DY$ ← using operator notation.

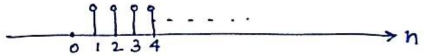
$\Rightarrow \quad Y - DY = X$, i/p can be determined from its o/p.

Example:

Y :

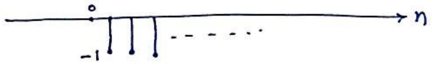


DY :



Right Shift (Delayed)

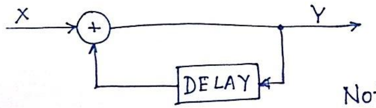
−DY :



Inverted

Y−DY :

unit sample sequence



Not imperative

$$Y = X + BY$$

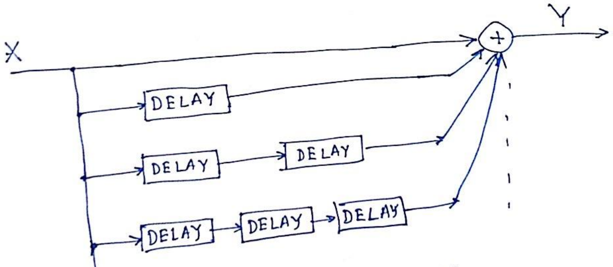
$$\Rightarrow \quad Y - BY = X$$

$$\Rightarrow \quad (1 - B)\,Y = X$$

$$\Rightarrow \quad Y = \left(\frac{1}{1-B}\right) X$$

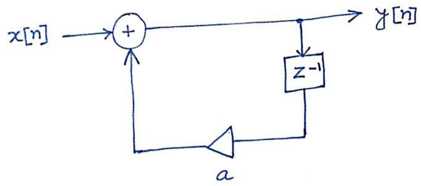
$$= \left(1 + B + B^2 + B^3 + \cdots\right) X$$

$$
\begin{array}{r|l}
1-B & 1 \qquad\qquad \left(1 + B + B^2 + \cdots\right. \\
 & \underline{1 - B} \\
 & \quad B \\
 & \quad \underline{B - B^2} \\
 & \qquad B^2 \\
 & \qquad \underline{B^2 - B^3} \\
 & \qquad\quad B^3
\end{array}
$$



Imperative

First-order recursive filter



Block diagram representation

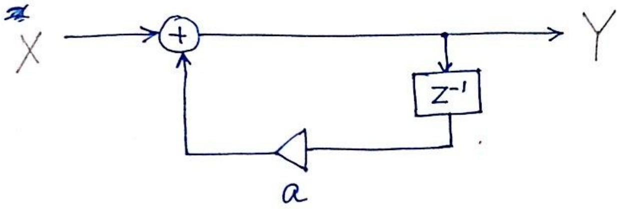
Difference equation:

$$y[n] = a \cdot y[n-1] + x[n]$$

The current value of the output is equal to the sum of the input and 'a' times the value of the previous output. As the current output is also a function of the previous outputs, it is called a recursive filter.

The term order is used to denote the minimum number of 'delays' and 'advances' that are necessary to implement a digital filter. Since, only one delay is necessary (here to implement), this is a first order filter.

First-order Recursive Filter



Operator approach:

$$Y = X + aDY$$

$$\Rightarrow \quad Y - aDY = X$$

$$\Rightarrow \quad \cancel{Y}\,(1 - aD)\,Y = X$$

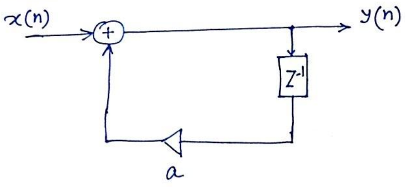
$$\Rightarrow \quad \frac{Y}{X} = \frac{1}{1 - aD}$$

Compare it with $H(z) = \dfrac{1}{1 - az^{-1}}$

System functional

$$D \equiv z^{-1}$$

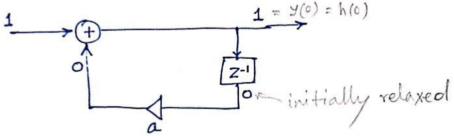
First-order Recursive Filter :



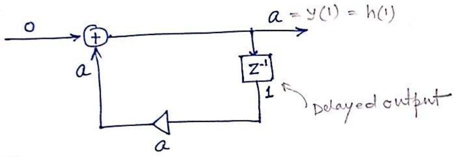
Informal Approach : (Sample by sample basis)

(Intuitive)

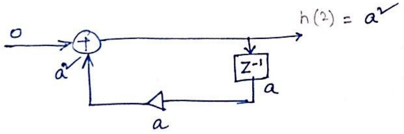
At n=0,



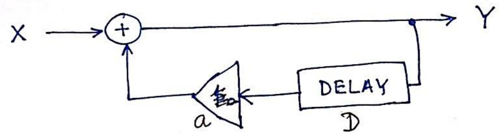
After clocking
↓
At n= 1,



After another clock
↓
At n=2,



Hence, $\boxed{h(n) = a^n u(n)}$

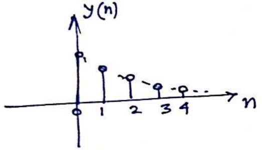


ii) $h(n) = a^n, \quad \text{if } n \geq 0;$

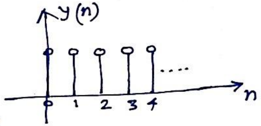
$\quad\quad\;\, = 0, \quad \text{otherwise}$

a is called the pole

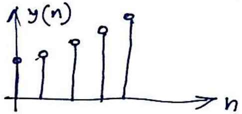
or, $\boxed{h(n) = a^n u(n)}, \quad \forall n$



$a = 0.5$



$a = 1$



$a = 1.2$

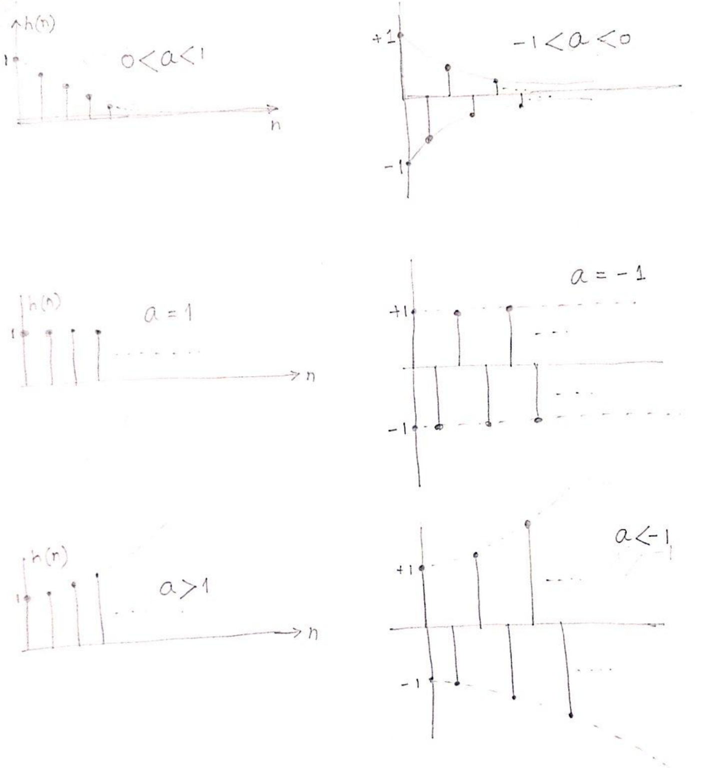
using operator notation we can write the following —

i) $X + aDY = Y$

$\Rightarrow X = (1 - aD)Y$

$\Rightarrow \boxed{\dfrac{Y}{X} = \dfrac{1}{1 - aD}}$ for 1st order filter

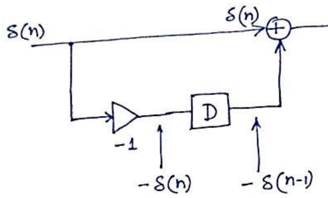
Sketch of $h(n) = a^n u(n)$:



Note: To have a stable filter "a" must be numerically less than '1', sign may be +ve or -ve.

To obtain impulse (unit sample) response of a filter from its block diagram by inspection

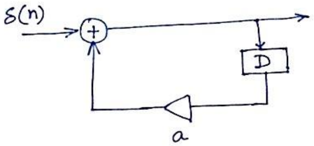
i) Non-recursive filter:



$h(n) = \delta(n) - \delta(n-1)$

$$\frac{Y}{X} = (1 - D)$$

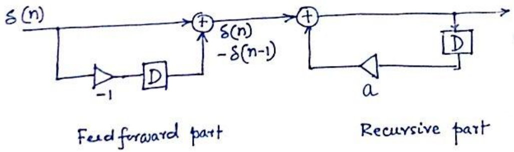
ii) Recursive first-order filter:



$h(n) = a^n u(n)$

$$\frac{Y}{X} = \frac{1}{1 - aD}$$

iii) (Combined) General first-order filter:

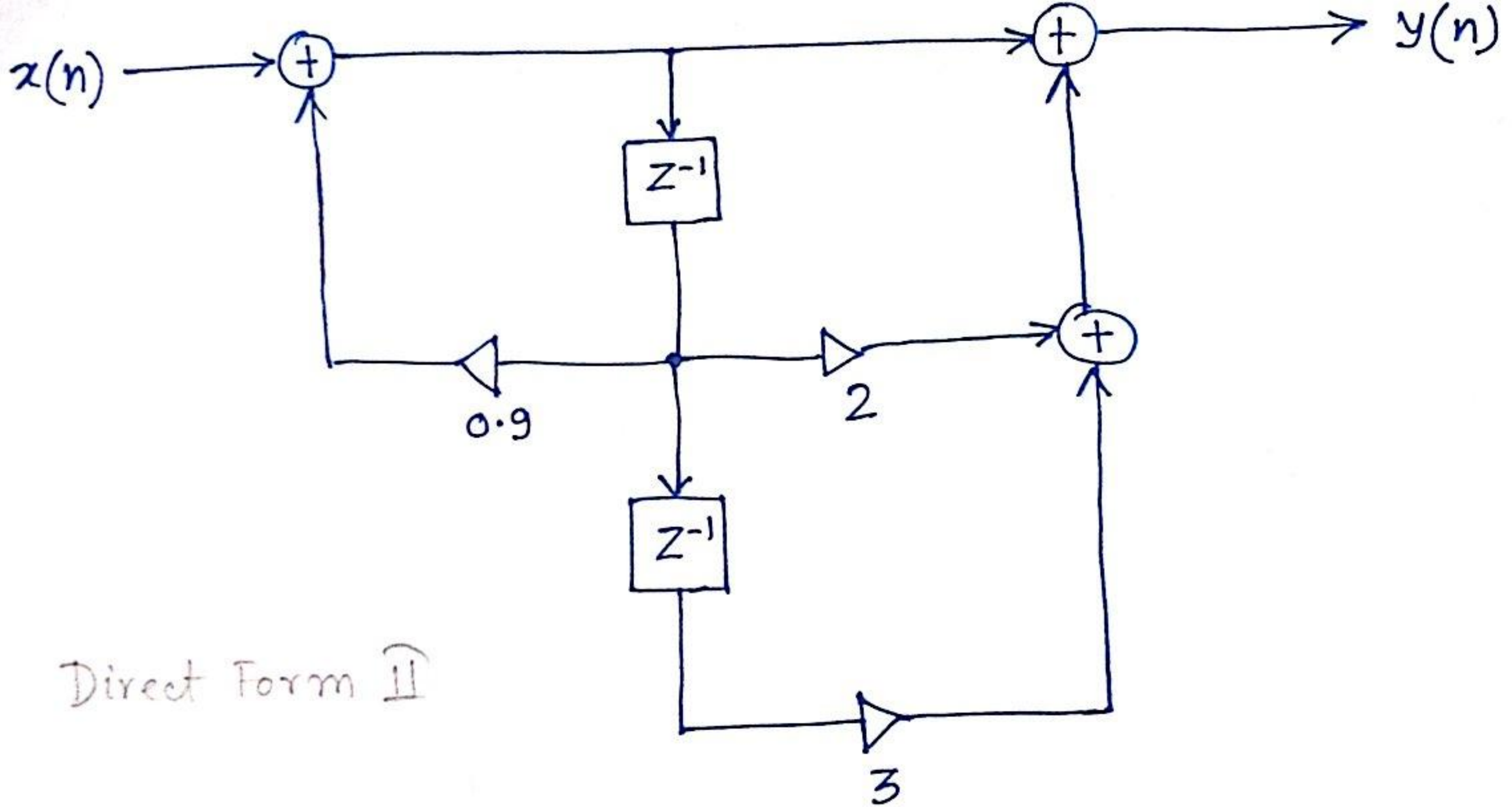


$h(n) = a^n u(n) - a^{n-1} u(n-1)$

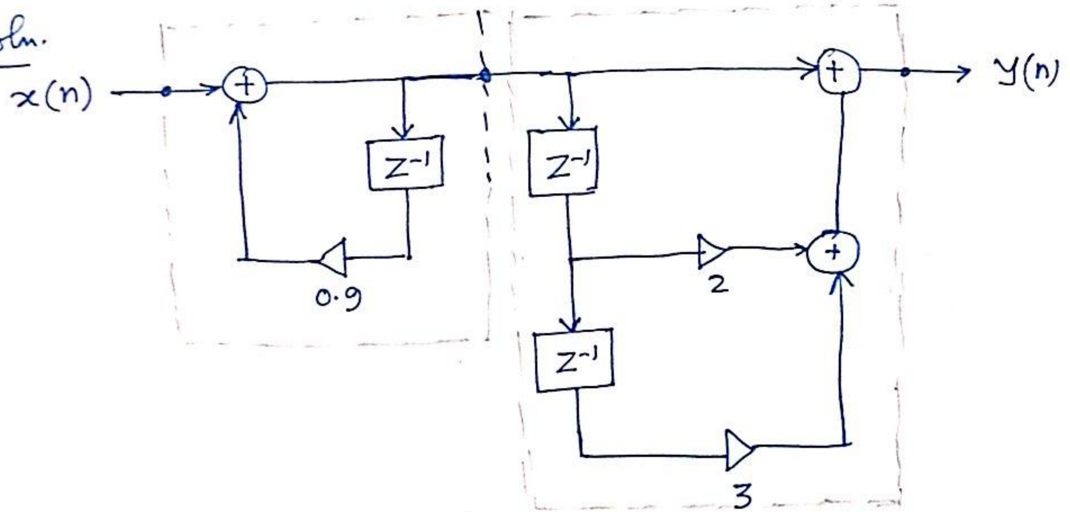
Filter (realization) structures

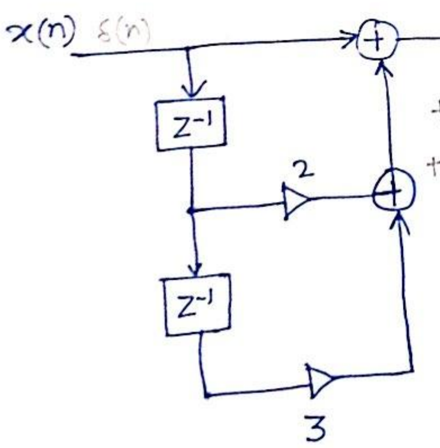
- Direct form I
- Direct form II
- Series (cascade) form
- Parallel form

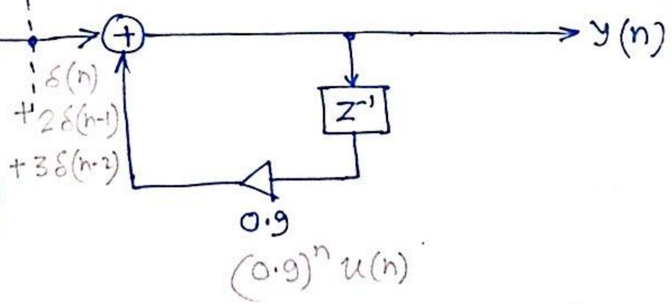
Prob. Obtain the impulse response of the following filter.



Soln.



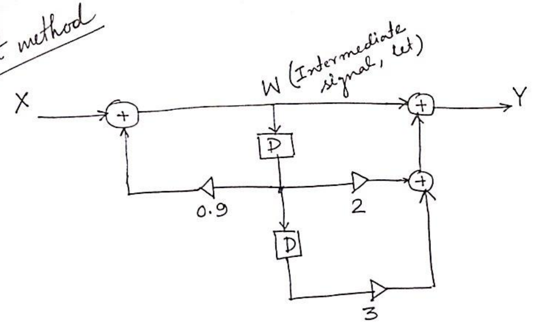




$$h(n) = (0.9)^n u(n) + 2(0.9)^{n-1} u(n-1) + 3(0.9)^{n-2} u(n-2)$$

Direct Form I

Alt method



Using operator notation,

$$0.9\,DW + X = W$$

$$\Rightarrow \quad X = (1 - 0.9D)\,W \qquad \text{—— (i)}$$

$$W + 2DW + 3D^2W = Y \qquad \text{—— (ii)}$$

$$\Rightarrow \quad (1 + 2D + 3D^2)\,W = Y$$

$$Y = (1 + 2D + 3D^2)\,W = (1 + 2D + 3D^2) \times \frac{X}{1 - 0.9D}$$

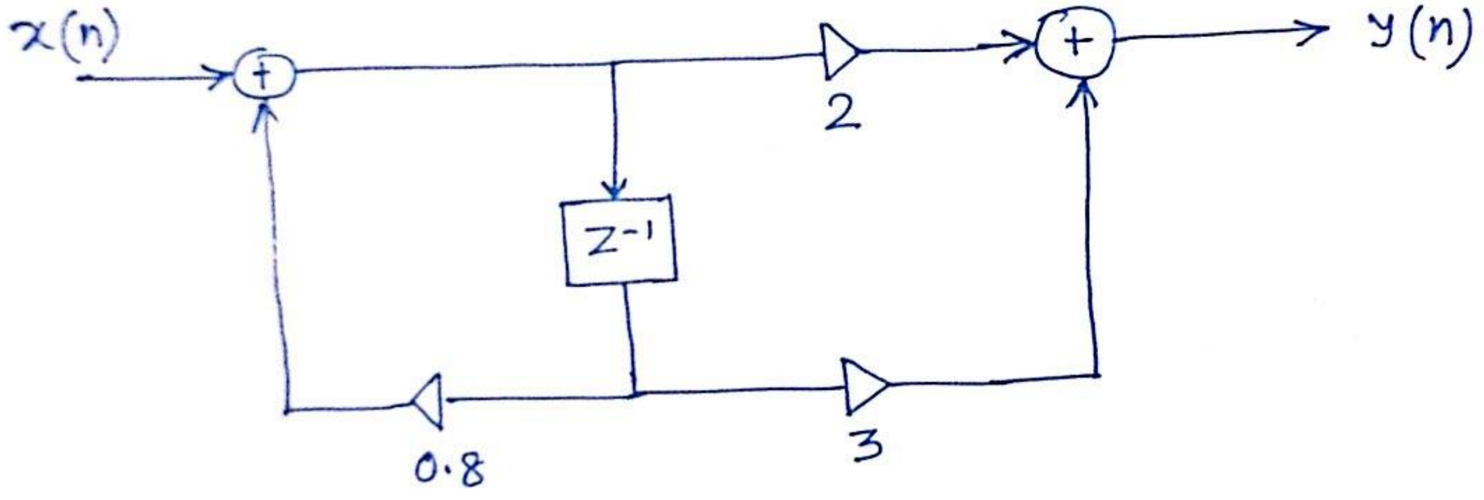
$$\Rightarrow \quad \frac{Y}{X} = \frac{1 + 2D + 3D^2}{1 - 0.9D} \qquad \Leftarrow \text{System functional}$$

$$\Rightarrow \quad (1 - 0.9D)\,Y = (1 + 2D + 3D^2)\,X$$

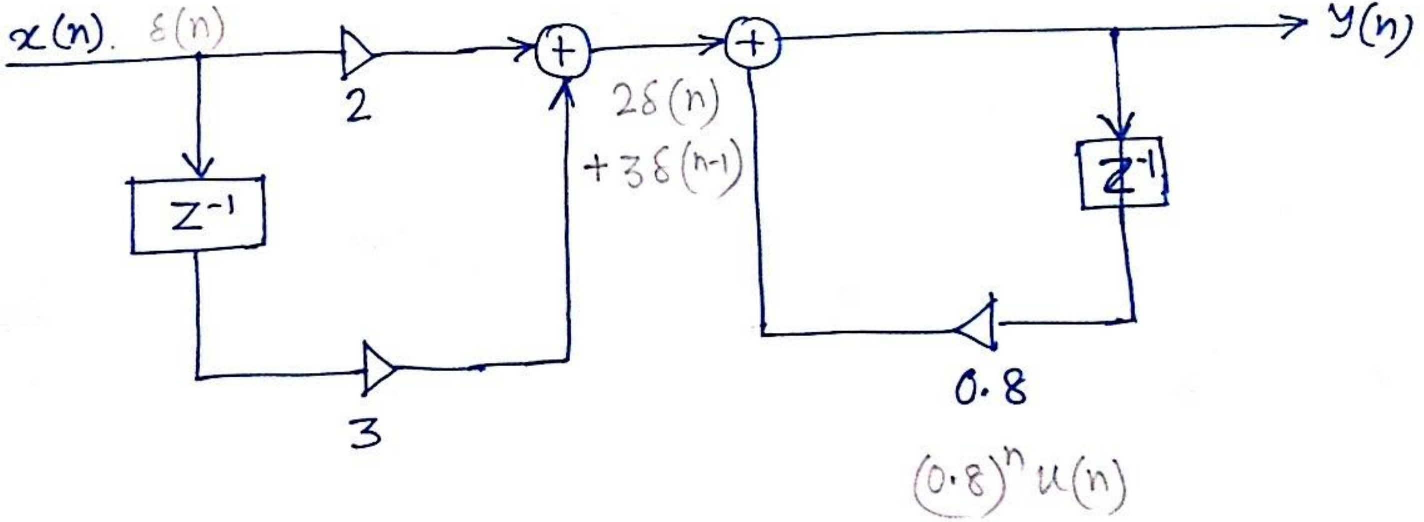
$$\Rightarrow \quad y(n) - 0.9\,y(n-1) = x(n) + 2x(n-1) + 3x(n-2)$$

$$\Rightarrow \quad y(n) = x(n) + 2x(n-1) + 3x(n-2) + 0.9\,y(n-1)$$

Prob. Obtain the impulse response of the following filter.



Soln.

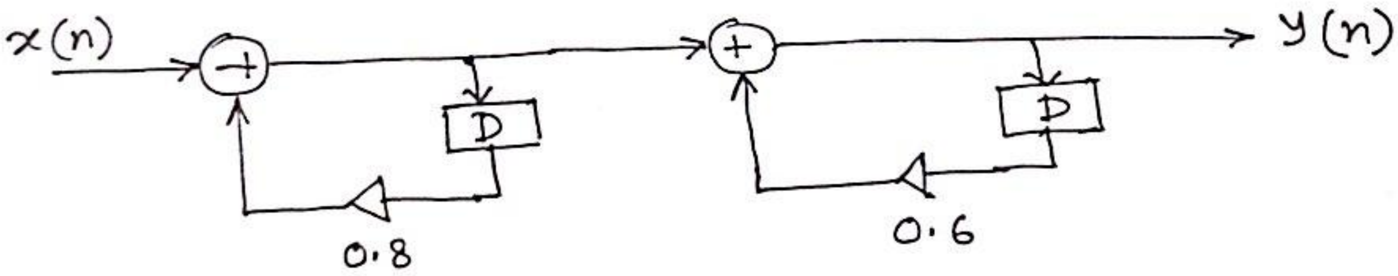


$$h(n) = 2(0.8)^n u(n) + 3(0.8)^{n-1} u(n-1)$$

Problem: Block diagram of a DT system is given below.

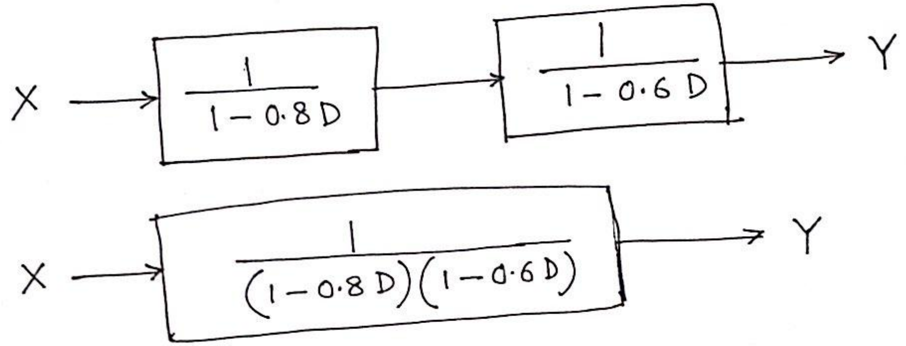
a) Determine the difference equation.

b) Obtain the impulse response.



Soln.

a)



$$\frac{Y}{X} = \frac{1}{(1-0.8D)(1-0.6D)}$$

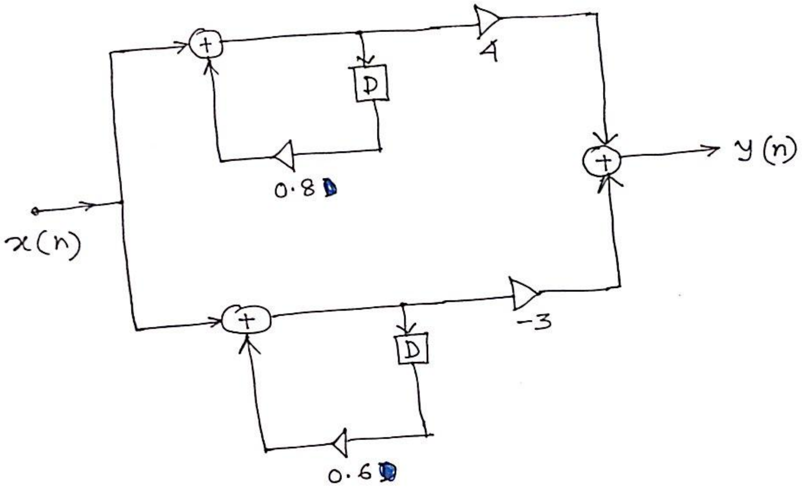
$$\Rightarrow \quad (1-0.8D)(1-0.6D)\,Y = X$$

$$\Rightarrow \quad (1 - 1.4D + 0.48D^2)\,Y = X$$

$$\Rightarrow \quad y(n) - 1.4\,y(n-1) + 0.48\,y(n-2) = x(n)$$

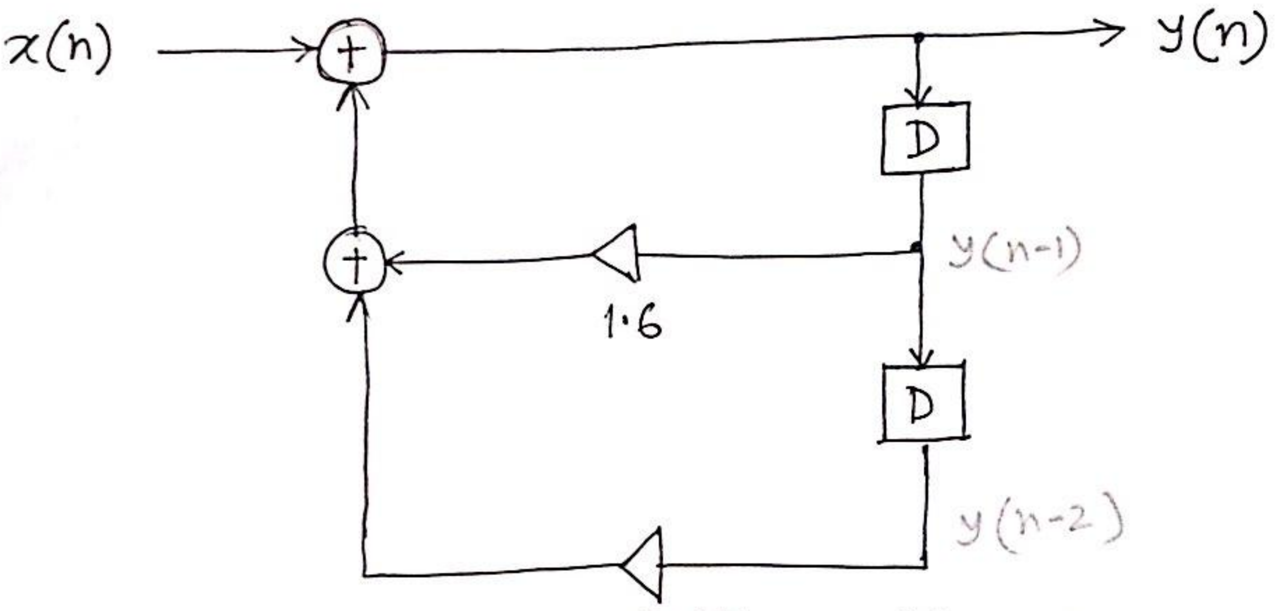
$$\Rightarrow \quad y(n) = x(n) + 1.4\,y(n-1) - 0.48\,y(n-2)$$

b) $$\frac{Y}{X} = \frac{1}{(1-0.8D)(1-0.6D)} = \frac{4}{1-0.8D} + \frac{-3}{1-0.6D}$$



$$h(n) = \left[4(0.8)^n - 3(0.6)^n\right] u(n) \qquad \forall n$$

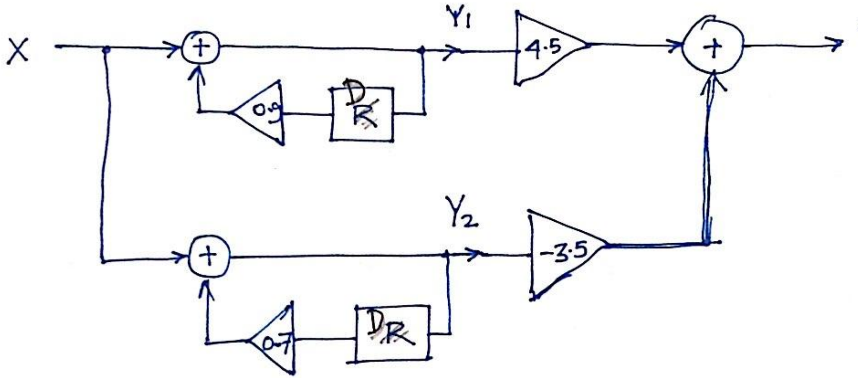
Second Order System



Block Diagram

$$\frac{Y}{X} = \frac{1}{(1-0.9D)(1-0.7D)} = \frac{4.5}{1-0.9D} - \frac{3.5}{1-0.7D}$$

$$= 4.5 \times \frac{1}{1-0.9D} + (-3.5) \times \frac{1}{1-0.7D}$$



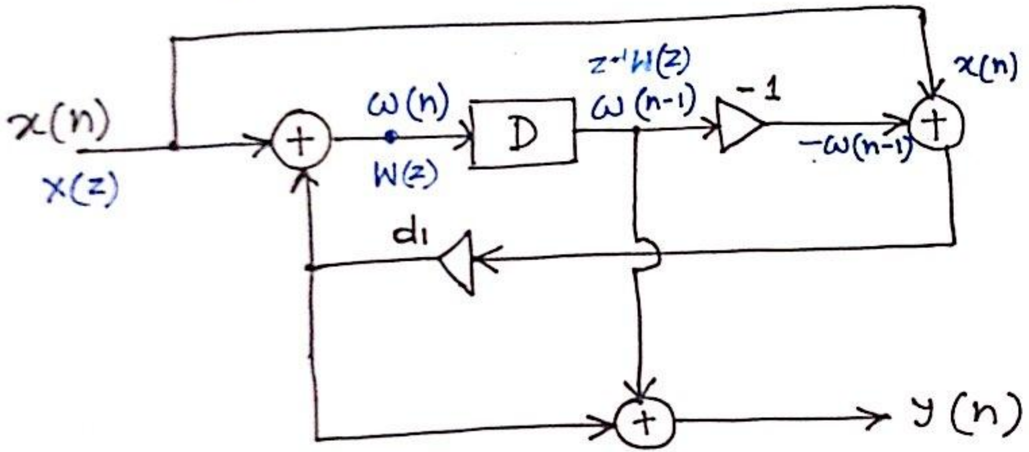
If $x(n) = \delta(n)$ then $y_1(n) = 0.9^n$

and $y_2(n) = 0.7^n$ for $n \geq 0$

Thus, $y(n) = 4.5 y_1(n) - 3.5 y_2(n)$

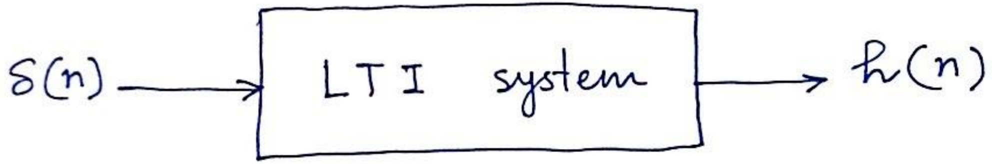
$$= 4.5(0.9)^n - 3.5(0.7)^n \quad \text{for } n \geq 0$$

Problem: Develop the relation between $y(n)$ and $x(n)$ from the following block diagram of a discrete-time system.



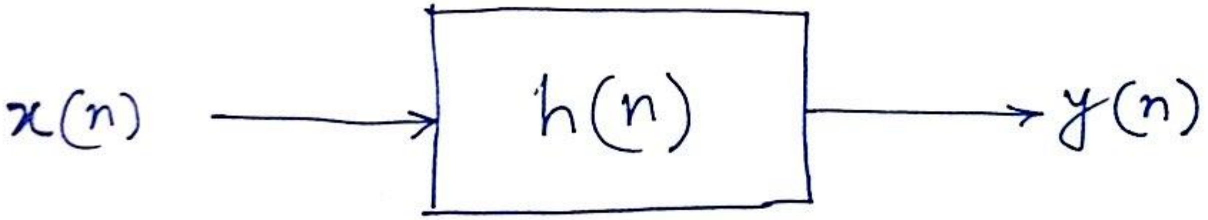
System representation using its impulse response :

A linear time-invariant (LTI) system can be completely described by its unit-impulse response, which is defined ~~by~~ as the system response due to the impulse input $\delta(n)$ with zero initial conditions.



unit-impulse response

With the obtained unit-impulse response $h(n)$, we can represent the linear time-invariant system.

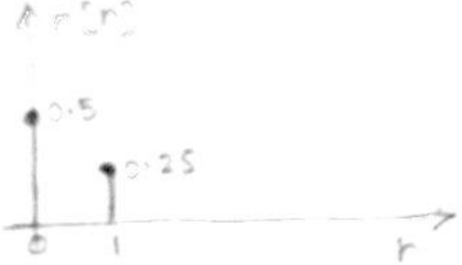


<u>system representation</u>

FIR System (FIR Filter)

b. $x(n) \longrightarrow \boxed{h(n) = 0.5\,\delta(n) + 0.25\,\delta(n-1)} \longrightarrow y(n)$

• h[n] contains finite number of terms and hence then it is called as finite impulse response (FIR) system



$$\Rightarrow h[n] = 0.5\,\delta[n] + 0.25\,\delta[n-1]$$

IIR System (IIR Filter)

$$h(n) = (0.25)^n u(n)$$

$$= \delta(n) + 0.25\,\delta(n-1) + (0.25)^2\,\delta(n-2) + \cdots$$

expressed in $\delta(n)$

b.

$$x(n) \longrightarrow \boxed{h(n) = \delta(n) + 0.25\,\delta(n-1) + (0.25)^2\,\delta(n-2) + \cdots} \longrightarrow y(n)$$

Infinite Impulse-response (IIR) system

as it contains infinite no. of terms.