Digital Signal Processing

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Response due to complex exponential

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = \sum_{K=-\infty}^{\infty} h(K) \cdot x(n-K)$$

$$= \sum_{K=-\infty}^{\infty} h(K) \cdot e^{j\omega_0 (n-K)}$$

$$= \sum_{K=-\infty}^{\infty} h(K) \cdot e^{j\omega_0 n} \cdot e^{-j\omega_0 K}$$

$$= \underline{\underline{e^{j\omega_0 n}}} \underbrace{\sum_{K=-\infty}^{\infty} h(K) \cdot e^{-j\omega_0 K}}_{H(e^{j\omega_0})}$$

same as the input complex exponential

constant, independant of n

$$H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega_0 k}$$

$$= \left| H(e^{j\omega_0}) \right| \cdot e^{j \angle H(e^{j\omega_0})}$$

phase

magnitude (gain)

$$y(n) = \left| H(e^{j\omega_0}) \right| \cdot e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}}$$

(Transfer function) Frequency response (function)

It defines the frequency transmission characteristics of the system. (filter)

ω : normalized radian frequency.

Soln:

$$h(n) = \{1, 2, 1\}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n) \cdot e^{-j\omega n}$$

$$= h(0) + h(1) \cdot e^{-j\omega} + h(2)\, e^{-j\omega 2}$$

$$= 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(\cancel{e^{j\omega}}\ 2 + 2\cos\omega\right)$$

$$= (2 + 2\cos\omega)\, e^{-j\omega}$$

Example

Consider the FIR filter defined by the impulse response

$$h(n) = -\delta(n) + 3\,\delta(n-1) - \delta(n-2)$$

$$\Rightarrow h(n) = \{-1,\ 3,\ -1\}$$

$$\quad\ \ \uparrow b_0 \quad \uparrow b_1 \quad \uparrow b_2$$

Diff. eqn.

$$\Rightarrow y(n) = -x(n) + 3\,x(n-1) - x(n-2)$$

$$\cancel{H(e^{j\omega}) = -1\cdot + 3e^{-j\omega\cdot 1} - e^{-j\omega 2}}$$

~~sample no. 0 sample no sample no. 2~~

~~(Directly, looking into the samples numbers)~~

$$H(e^{j\omega}) = \sum_{n=0}^{2} h(n)\cdot e^{-j\omega n}$$

$$= h(0)\cdot e^{-j\omega\cdot 0} + h(1)\cdot e^{-j\omega\cdot 1} + h(2)\cdot e^{-j\omega\cdot 2}$$

$$= -1 + 3e^{-j\omega} - e^{-j2\omega}$$

Prob. Frequency response of a filter is given by the equation

$$H(e^{j\omega}) = e^{-j\omega}(3 - 2\cos\omega)$$

Find the difference equation (of the filter).

Soln.

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$H(e^{j\omega}) = e^{-j\omega}\left(3 - 2 \times \frac{e^{j\omega} + e^{-j\omega}}{2}\right)$$

$$= e^{-j\omega}\left(3 - e^{j\omega} - e^{-j\omega}\right)$$

$$= 3e^{-j\omega} - e^{0} - e^{-j\omega 2}$$

$$= -1 + 3e^{-j\omega\cdot 1} - e^{-j\omega\cdot 2}$$

$$h(n) = \{\underset{b_0}{-1}, \underset{b_1}{3}, \underset{b_2}{-1}\}$$

$$y(n) = -x(n) + 3x(n-1) - 1.x(n-2)$$

Problem: Show that the frequency response of an FIR filter with coefficients $\{b_k\} = \{1, -2, 4, -2, 1\}$ can be expressed as

$$H(e^{j\omega}) = \left[4 - 4\cos\omega + 2\cos(2\omega)\right] e^{-j.2\omega}$$

Prob. Consider the complex input $x(n) = 2e^{j\pi/4}e^{j\pi . n/3}$. If this signal is input to the following system whose $h(n) = \{1, 2, 1\}$; Find the output of the system.

$$x(n) = 2e^{j\pi/4}.e^{j\pi . n/3} \longrightarrow \underset{h(n) = \{1,2,1\}}{\boxed{h(n)}} \longrightarrow ?\ y(n)$$

Soln.

Soln:

$$h(n) = \{1, 2, 1\}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n) \cdot e^{-j\omega n}$$

$$= h(0) + h(1) \cdot e^{-j\omega} + h(2)\, e^{-j\omega 2}$$

$$= 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(\cancel{e^{j\omega}}\ 2 + 2\cos\omega\right)$$

$$= (2 + 2\cos\omega)\, e^{-j\omega}$$

Since $(2+2\cos\omega) \geq 0$ for $-\pi < \omega \leq \pi$,

the magnitude is $|H(e^{j\omega})| = 2+2\cos\omega$

and the phase is $\angle H(e^{j\omega}) = -\omega$

P.T.O

$$\left|H(e^{j\omega})\right|_{\omega=\pi/3} = 2 + 2\cos\omega \quad 3$$

$$= 2 + 2\cos\pi/3 = 2 + 2\times\frac{1}{2}$$

$$\angle H(e^{j\omega})\Big|_{\omega=\pi/3} = -\omega = -\pi/3$$

$$H(e^{j\omega})\Big|_{\omega=\pi/3} = 3\cdot e^{-j\cdot\pi/3}$$

$$\therefore\ y(n) = \underline{3}\cdot 2e^{j\pi/4}\cdot e^{j(\pi\cdot n/3 \underline{\underline{- \pi/3}})}$$

$$= 6\cdot e^{j\cdot\pi/4}\cdot e^{j\pi(n-1)/3}$$

The output is equal to the input multiplied by 3 and the phase shift corresponds to a delay of one sample.

Response to a sinusoidal signal

$$x(n) = A_0 + A_1 \cos(\omega_1 n + \phi_1) \longrightarrow \boxed{\; h(n) \;} \longrightarrow ?$$

$$x(n) = A_0 + A_1 \cos(\omega_1 n + \phi_1)$$

$$= A_0 \cdot e^{j.0n} + A_1 \left[\frac{e^{j(\omega_1 n + \phi_1)} + e^{-j(\omega_1 n + \phi_1)}}{2} \right]$$

$$= A_0 \cdot e^{j.0.n} + \frac{A_1}{2} \cdot e^{j(\omega_1 n + \phi_1)} + \frac{A_1}{2} \cdot e^{-j(\omega_1 n + \phi_1)}$$

$$y(n) = H(e^{j.0}) A_0 e^{j0.n} + H(e^{j\omega_1}) \cdot \frac{A_1}{2} \cdot e^{j(\omega_1 n + \phi_1)}$$

$$+ H(e^{-j\omega_1}) \frac{A_1}{2} \cdot e^{-j(\omega_1 n + \phi_1)}$$

$$= H(e^{j.0}) A_0 e^{j.0.n} + |H(e^{j\omega_1})| e^{j\angle H(e^{j\omega_1})} \cdot \frac{A_1}{2} \cdot e^{+j(\omega_1 n + \phi_1)}$$

$$+ |H(e^{j\omega_1})| e^{-j.\angle H(e^{-j\omega_1})} \cdot \frac{A_1}{2} \cdot e^{-j(\omega_1 n + \phi_1)}$$

$$= H(e^{j.0}) A_0 \cdot e^{j.0.n} + |H(e^{j\omega_1})| A_1 \cos(\omega_1 n + \phi_1 + \angle H(j\omega_1))$$

Prob. For the FIR filter with coefficients $\{b_k\} = \{1, 2, 1\}$ find the output when the input is

$$x(n) = 2\cos\left(\frac{\pi}{3}\cdot n - \frac{\pi}{2}\right)$$

Soln:

$$\underbrace{2}_{\text{amplitude}}\cos\left(\frac{\pi}{3}\cdot n - \frac{\pi}{2}\right) \longrightarrow \boxed{h(n) = \{1,\ 2,\ 1\}} \longrightarrow \ ?$$

Soln.

$$h(n) = \{1, 2, 1\}$$

$$H(e^{j\omega}) = (2 + 2\cos\omega)\, e^{-j\omega}$$

$$H(e^{j.\pi/3}) = 3.\, e^{-j.\pi/3}$$

← phase

↑ magnitude (gain)

$$\therefore \quad y(n) = 6 \cos\left(\pi/3 \cdot n - \pi/2 - \pi/3\right)$$

$$= 6 \cos\left[\pi/3\,(n-1) - \pi/2\right]$$

The magnitude of the frequency response multiplies the amplitude of the cosine signal and the phase angle of the frequency response adds to phase of the cosine signal.

Prob. For the FIR filter with coefficients $b_k = \{1, 2, 1\}$, find the output when the input is

$$x(n) = 4 + 3\cos\left(\frac{\pi}{3}\cdot n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}\cdot n\right)$$

Soln.

$$h(n) = \{1, 2, 1\}$$

$$H(e^{j\omega}) = (2 + 2\cos\omega) \cdot e^{-j\omega}$$

Three components

$$H(e^{j \cdot 0}) = (2 + 2\cos 0)\, e^{-j \cdot 0} = 4$$

$$H(e^{j \cdot \pi/3}) = 3 \cdot e^{-j \cdot \pi/3}$$

$$H(e^{j \cdot \frac{7\pi}{8}}) = 0.1522\, e^{-j \cdot 7\pi/8}$$

$$\therefore \quad y(n) = 4 \cdot 4 + 3 \cdot 3 \cos\left(\pi/3 \cdot n - \pi/2 - \pi/3\right)$$

$$+ 0.1522 \times 3 \cos\left(\frac{7\pi}{8} \cdot n - \frac{7\pi}{8}\right)$$

$$= 16 + 9 \cos\left(\pi/3\,(n-1) - \pi/2\right)$$

$$+ \underbrace{0.4566}_{\text{very low value.}} \cos\left(\frac{7\pi}{8}(n-1)\right)$$

$\Rightarrow$ essentially filtered out

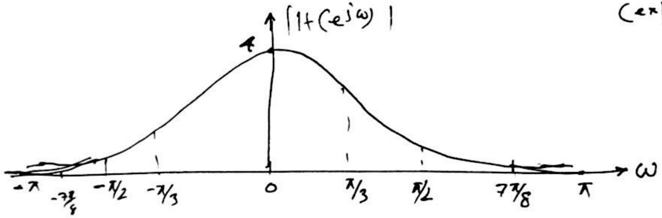
. Low pass Filter: $\longrightarrow \boxed{h(n) = \{1, 2, 1\}} \longrightarrow$

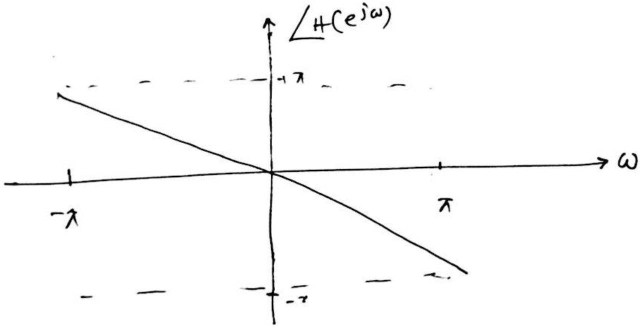
$$H(e^{j\omega}) = 1 + 2 \cdot e^{-j\omega} + e^{-j2\omega} = (2 + 2\cos\omega)\, e^{-j\omega}$$

$$|H(e^{j\omega})| = 2 + 2\cos\omega$$

$$\angle H(e^{j\omega}) = -\omega$$

slope of -1
=> delay of 1 sample
(experienced by all ~~samples~~ frequencies).





Conjugate Symmetry

For real valued sequence, $h(n)$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} h(n) \cdot \cos\omega n}_{H_R(e^{j\omega})} + j\,(-1) \underbrace{\sum_{n=-\infty}^{\infty} h(n) \sin\omega n}_{H_I(e^{j\omega})}$$

$$H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{+j\omega n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} h(n) \cos\omega n}_{H_R(e^{-j\omega})} + j \cdot \underbrace{\sum_{n=-\infty}^{\infty} h(n) \sin\omega n}_{H_I(e^{-j\omega})}$$

So, $H^*(e^{j\omega}) = H(e^{-j\omega})$

Symmetry of the magnitude respone

$$|H(e^{j\omega})| = \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$$

$$|H(e^{-j\omega})| = \sqrt{H_R^2(e^{-j\omega}) + H_I^2(e^{-j\omega})}$$

$$= \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$$

$$= |H(e^{j\omega})|$$

$$\Rightarrow \quad |H(e^{-j\omega})| = |H(e^{j\omega})|$$

Hence, magnitude response is <u>even symmetric</u>.

Symmetry of the phase response

$$\angle H(e^{j\omega}) = \tan^{-1} \frac{-\sum_{n=-\infty}^{\infty} h(n) \cdot \sin\omega n}{\sum_{n=-\infty}^{\infty} h(n) \cdot \cos\omega n}$$

$$= -\tan^{-1} \frac{\sum_{n=-\infty}^{\infty} h(n) \cdot \sin\omega n}{\sum_{n=-\infty}^{\infty} h(n) \cdot \cos\omega n}$$

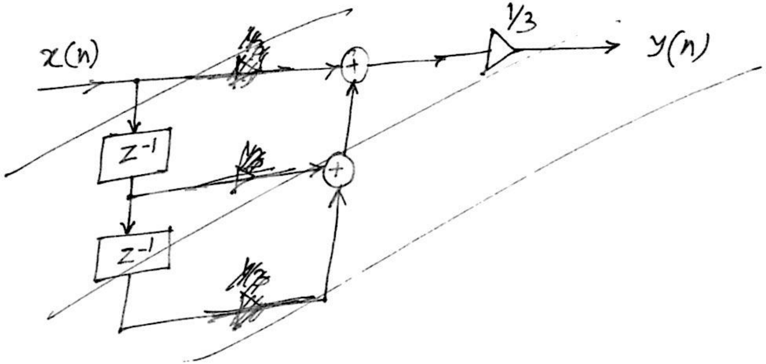
$$\angle H(e^{-j\omega}) = \tan^{-1} \frac{\sum_{n=-\infty}^{\infty} h(n) \cdot \sin\omega n}{\sum_{n=-\infty}^{\infty} h(n) \cdot \cos\omega n}$$

$$\Rightarrow \quad \angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$$

Hence, phase response is odd symmetric.

3 - point moving average filter

$$y(n) = \frac{x(n) + x(n-1) + x(n-2)}{3}$$



$$h(n) = \frac{1}{3}\delta(n) + \frac{1}{3}\delta(n-1) + \frac{1}{3}\delta(n-2)$$

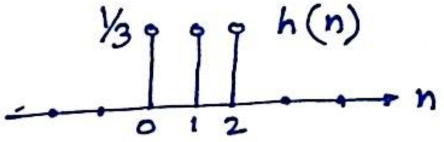
$$= \left\{ \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3}, \frac{1}{3} \right\}$$

$$h(n) = \frac{1}{3} \quad \text{for } 0 \leq n \leq 2$$

$$= 0 \quad \text{Otherwise}$$

Frequency response of (causal) 3-point moving average filter:

$$h(n) = \begin{cases} 1/3 & \text{for } 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$



Then, $$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n} = \frac{1}{3}\sum_{n=0}^{2} e^{-j\omega n}$$

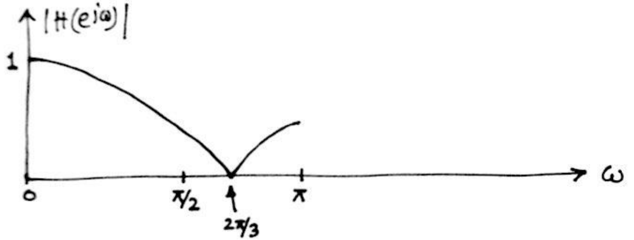
$$= \frac{1}{3}\left[1 + e^{-j\omega} + e^{-j\omega 2}\right]$$

$$= \frac{1}{3} e^{-j\omega}\left[e^{j\omega} + 1 + e^{-j\omega}\right]$$

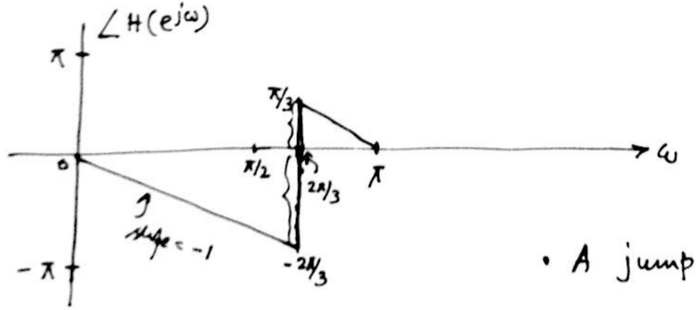
$$= \frac{1}{3}(1 + 2\cos\omega)\, e^{-j\omega} = A(e^{j\omega}) \cdot e^{-j\omega}$$

$$= \frac{1}{3}\left(1+2\cos\omega\right)e^{-j\omega} = A(e^{j\omega})\cdot e^{-j\omega}$$

$$|H(e^{j\omega})| = |A(e^{j\omega})| = \frac{1}{3}\left|(1+2\cos\omega)\right|$$

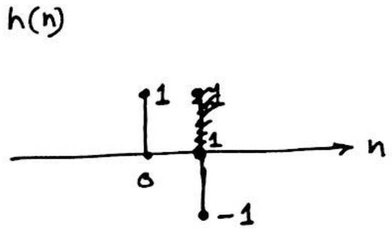


$$\angle H(e^{j\omega}) = \begin{cases} -\omega & \text{when } A(e^{j\omega}) > 0 \quad \text{i.e. } 0 \le \omega \le 2\pi/3 \\ -\omega \pm \pi & \text{when } A(e^{j\omega}) < 0 \quad \text{i.e. } 2\pi/3 < \omega \le \pi \end{cases}$$



- A jump of $+\pi$ occurs

First - difference System:



- Difference eqn: $y(n) = x(n) - x(n-1)$

$$h(n) = \delta(n) - \delta(n-1)$$

- Impulse Response: $h(n) = \{1, -1\}$

- Frequency response of this LTI system is

$$H(e^{j\omega}) = \sum_{n=0}^{1} h(n) \cdot e^{-j\omega n} = h(0) \cdot e^{-j\omega \cdot 0} + h(1) \cdot e^{-j\omega \cdot 1}$$

$$= h(0) + h(1) \cdot e^{-j\omega}$$

$$= 1 - e^{-j\omega}$$

$$= e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= e^{-j\omega/2} \cdot 2j \cdot \sin(\omega/2)$$

$$= \underbrace{2\sin(\omega/2)} \cdot \underbrace{e^{+j(\pi/2 - \omega/2)}} \quad \text{as} \rightarrow j = e^{j\pi/2}$$

$$= \underbrace{2\sin(\omega/2)}\cdot\underbrace{e}^{+j(\pi/2-\omega/2)}$$ as →

• Magnitude:

$$|H(e^{j\omega})| = 2\,|\sin(\omega/2)|$$

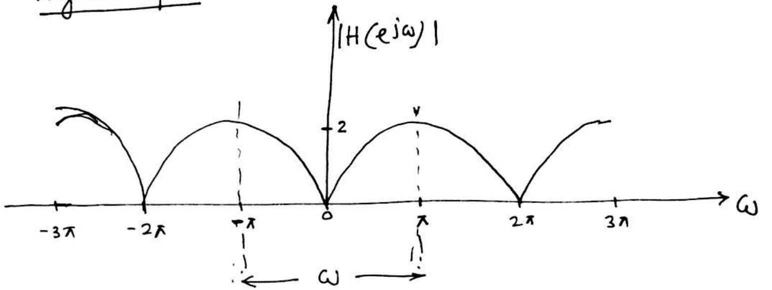
• Phase:

$$\angle H(e^{j\omega}) = \begin{cases} \pi/2 - \omega/2, & 0 < \omega < \pi \end{cases}$$

consider only this because we need only $0 < \omega < \pi$ range.

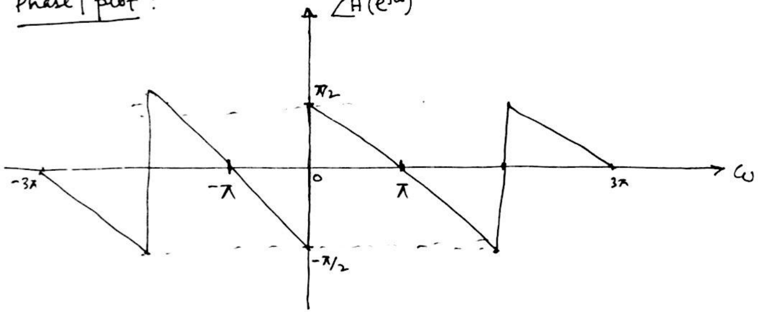
(freq^n response)

Magnitude plot :



(freq^n response)

Phase plot :



Conclusions :

- Highpass Filter : HPF

- DC is eliminated : This could be told long back.

$$h(n) = \{1, -1\}$$

$$\text{dc gain} = 1 + (-1) = 0$$

Periodicity

The frequency response is a periodic function of the continuous-valued variable ω, with period 2π.

$$H\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j(\omega+2\pi k)\cdot n}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n} \cdot e^{-j \cdot 2\pi k \cdot n}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n} \qquad \text{as } e^{-j2\pi k} = 1$$

$$= H\left(e^{j\omega}\right)$$

First difference removes DC

Prob. $$x(n) = 4 + 2\cos(0.3\pi n - \pi/4)$$

$$y(n) = x(n) - x(n-1)$$

$$= 4 + 2\cos(0.3\pi n - \pi/4) - 4 - 2\cos(0.3\pi(n-1) - \pi/4)$$

$$= 2\cos(0.3\pi n - \pi/4) - 2\cos(0.3\pi n - 0.55\pi)$$

Alt. $$H(e^{j\omega}) = 2\sin(\omega/2)\, e^{j(\pi/2 - \omega/2)}$$

$$y(n) = 4H(e^{j0}) + 2\left|H(e^{j0.3\pi})\right| \cos\left(0.3\pi n - \pi/4 + \angle H(e^{j0.3\pi})\right)$$

$$H(e^{j0}) = 0$$

$$H(e^{j0.3\pi}) = 2j\sin(0.3\pi/2)\, e^{-j.03\pi/2} = 0.908\, e^{j(\pi/2 - 0.15\pi)}$$

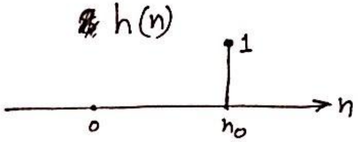
$$y(n) = (0.908)(2)\cos\left(0.3\pi n - \pi/4 + \pi/2 - 0.3\pi/2\right)$$

$$= 1.816\cos(0.3\pi n + 0.1\pi)$$

Delay System :

· **Diff. eqn.** $\quad y(n) = x(n-n_0)$

· **Impulse response :** $\quad h(n) = \delta(n-n_0)$



It has only one non-zero filter coefficient, $b_{n_0} = 1$, so its frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n} = e^{-j\omega n_0} = \underset{\text{Mag}}{1} \cdot e^{\underbrace{-j\omega n_0}_{\text{Phase}}}$$

$$|H(e^{j\omega})| = 1$$

· Magnitude response is constant equal to 1 for all frequencies

and $\angle H(e^{j\omega}) = -n_0\omega$

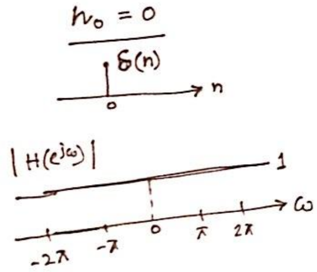
Hence, · phase response is linear with frequency, equal to $-n_0\omega$

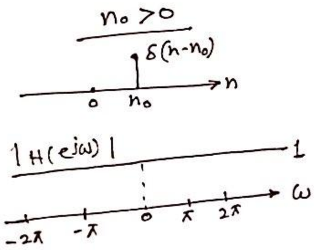
$$|H(e^{j\omega})| = 1$$

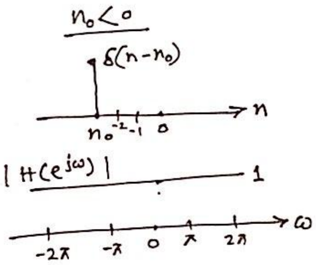
- Magnitude response is constant equal to 1 for all frequencies

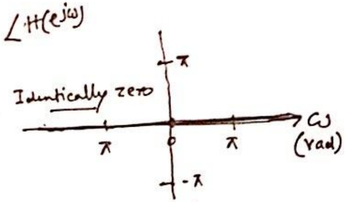
and $\angle H(e^{j\omega}) = -n_0\omega$

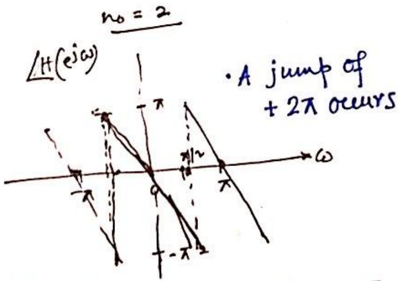
Hence, phase response is linear with frequency, equal to $-n_0\omega$

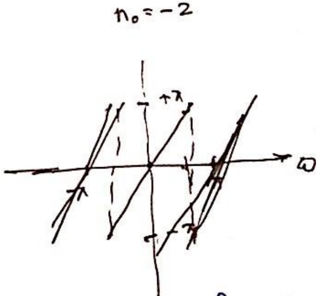












Principal value of the phase lying in the range $[-\pi, \pi]$

- A jump of -2π occurs

Phase Jumps:

There are two occas~~s~~ions (in last two examples) for which the phase function experiences discontinuities, or jumps.

i) A jump of $\pm 2\pi$ occurs to maintain the phase function within the principal value range of $[-\pi, \pi]$.

ii) A jump of $\pm \pi$ occurs when $A(e^{j\omega})$ undergoes a change in sign.

amplitude function

The sign of the jump is chosen such that the resulting phase function is odd – and, after the jump, lies in the range $[-\pi, \pi]$.