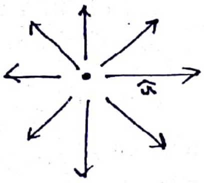
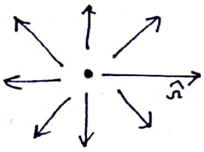
Larmor's Formula

Larmor's formula describe about the radiated power from an accelerated charge.

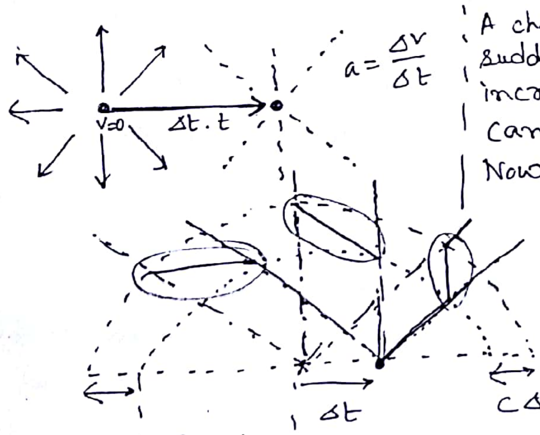


If a charge is placed as static then $\vec{F} = q\vec{E}$ where $\vec{E}$ is radially outward from the point charge. $\vec{E} = E(r)\hat{r}$
But as no oscillation and hence E is also static, not charge.



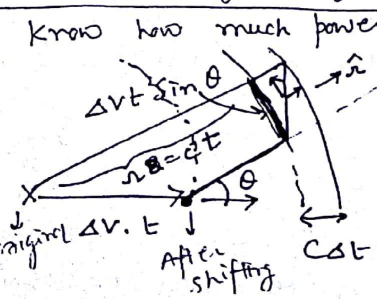
Now if this static charge is in motion then it has associated with another field and known as magnetic field. E is always radial in direction
$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
But as the motion is uniform hence E field always moving radially outward and also it does not have any disturbances. A person sitting on $\hat{r}$ at certain distance always feel E field coming out after some time due to the fact that no velocity be larger than light. So what you feel or observe is basically coming some amount earlier. But as this field also uniform so no planewave is generating.

So to have a radiation we need to have acceleration on charge, then only we can have some plane wave ($E \perp B$) coming out of charge.



A charge at rest producing E field. Now when we apply a suddent force and then stop then the charge will have a increase in velocity Δv for a time t. So the acceleration can be described as $\Delta v/\Delta t$. Now we can redraw the field bevion in a different way

So at far distance the field are always anticipating the information to come out from previous center as the information has not have enough time to travel to observe. Now internedly there is a new electric field orientation over charge. which is shifted due to acceleration. This electric field has a new center. Now we have two different electric field. But from conservation of law electric field always has continuity. It ~~always~~ So we need to have some transition. See that transition is $\perp^r$ to acceleration.. Now from this we can see that if the information is travelling outward $\boxed{\text{with a velocity close to light } c}$ and distance it varies is $c\Delta t$. Finally for a shift of charge position Δt the, now we can say that there is a disturbance in E field that moving at $c\hat{r}$. Also E field has not $\parallel^r$ to $\hat{r}$ now as previous. It has a two component one $\perp^r$ & $\parallel^r$. Similarly B field also two componets. Now we have a disturbance or a planewave moving outward. Note that after the initial acceleration if the charge moves with uniform velocity then there will be no further disturbance moving outward. Charge will go with uniform velocity. Similarly E field will move uniformly shifted. So no further kink / Disturbance. The disturbance created will travel with a speed of light at different time. Also the disturbance magnitude is minimum along the path of charge motion and maximum $\perp^r$ to charge motion. Now we need to know how much power is radiating out we need to know.



$E_\perp$?? ← important

$E_\parallel$?? ← Quasi

We need to find out the ratio between this in terms of slope.

$$\frac{E_\perp}{E_{\parallel}} = \frac{\Delta v \cdot t \sin\theta}{c\,\Delta t} = \frac{\Delta v}{\Delta t} \cdot \frac{t}{c} \sin\theta =$$

t can be expressed interms of radius r and velocity c

$$\therefore \frac{E_\perp}{E_{\parallel}} = acc \times \frac{r}{c^2} \sin\theta$$

Now from Coulomb's law we know radial component of E field as

$$\boxed{E_r = \frac{q}{r^2} \frac{1}{4\pi\epsilon_0}}$$

$$\therefore \boxed{E_\perp = \frac{a \cdot q}{4\pi\epsilon_0 c^2 r} \sin\theta}$$

From this we can observe that if $\theta = 0°$ $E = 0$ and for $\theta = 90°$

$$E = \frac{aq}{c^2 r}$$

It means that along the acceleration of charge E field has no variation or no disturbance but at $\perp^r$ direction of charge path it has highest disturbance.

Before Calculating the exact power or Radiated Energy we need to discuss two interesting situation.

There are two kind of system where velocity of charge can be described.

→ Non - Relativistic. velocity → Cyclotron motion where velocity is $\ll c$

It is a special case of Relativistic condition.

→ Relativistic velocity → Synchrotron radiation → where $v \approx c$.

In a Relativistic form the momentum is described as

$\gamma m v$ where γ is Lorentz Factor. γ is a term by whi time, length and relativistic mass can be related while the object is moving.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

v → relative velocity for an observer.
c → velocity of light.

If $v \ll c$ then $\gamma = 1$ which is non-relativistic condition.

Relativistic Mass → γm
Relativistic momentum $\gamma m v$

Now for a ~~Relati~~ non-relativistic case we got the above equation of $E_\perp$.
also $a = \frac{\Delta v}{\Delta t}$. Now See the interesting phenomenon of E_r and $E_\perp$. Again $E_r \neq f^n(a)$ but $E_\perp = f^n(a)$. So as $E_r \propto \frac{1}{r^2}$ and $E_\perp \propto \frac{1}{r}$. $E_\perp$ is always lossing less strength than E_r. So when $r \uparrow\uparrow$ & the time passes. $E_r \downarrow\downarrow$ and can be neglected. So now only $E_\perp$ is left. $E_\perp$ is dependent on θ. When $\theta = 90°$ $E_\perp$ is max and $\theta = 0°$ $E_\perp$ is minimum. So it means there is no EM propagation along the axis of charge motion.

Now For only E field Energy density is $\frac{1}{2}\epsilon_0 E^2$ and for only B field $\frac{1}{2\mu_0}B^2$

For EM field $\boxed{\eta = \eta_E + \eta_B}$ → $\epsilon_0 E^2$

we can say $\eta = \epsilon_0 \left(\frac{q a \sin\theta}{4\pi\epsilon_0\, r c^2}\right)^2 \Rightarrow \propto \frac{a^2}{r^2}$

Now from total energy ~~dens~~ density must be associated with sum of It means total energy density must be associated with sum of transverse field energy residing within a spherical volume of radius $r = ct$.

Now as the volume of sphere increases r^2 rate and η decreases with $\frac{1}{r^2}$ rate

So when a charge particle accelerate it losses its energy density to its surrounding In an amount $\propto a^2$.

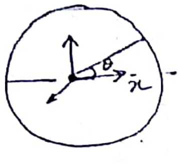
$E_\perp = \dfrac{q\, a \sin\theta}{4\pi \epsilon_0 c^2 r}$ & Energy per unit volume stored in E-field is

$$\eta_E = \frac{\epsilon_0}{2} \vec{E}^2$$

$\Rightarrow$ Energy per/volume $= \dfrac{q^2 a^2 \sin^2\theta}{32\pi^2 \epsilon_0 c^4 r^2}$

Now as there is a variation of η_E w.r.t θ hence it is better to describe the same with average equation.

Now assume Rectangular coordinate system at center of sphere with x along the particles original direction of motion



Now for any point (x, y, z) on spherical cell

$\cos\theta = \dfrac{x}{r}$ and $\sin^2\theta = 1 - \cos^2\theta = 1 - \dfrac{x^2}{r^2}$

Now we need to average out values at any points w.r.t center at origin.

We can say $x^2 = y^2 = z^2$ or Avg x^2 = Avg y^2 = Avg z^2

$\therefore$ Avg $x^2 = \frac{1}{3}(x^2 + y^2 + z^2) \approx \frac{1}{3} r^2$ and r is constant over the whole spherical shell.

We can write → Imagine any point in circle of Rad (r) can be represented by x, y, z

$\therefore \sin^2\theta = 1 - \dfrac{r^2}{3r^2} = \dfrac{2}{3}$

$$\therefore \boxed{\eta_E = \frac{q^2 a^2}{32\pi^2 \epsilon_0 c^4 r^2} \times \frac{2}{3} = \frac{q^2 a^2}{48\pi^2 \epsilon_0 c^4 r^2}}$$

Now total energy for E field we need to multiply the η_E with volume.

Avg Surface area of the shell is $4\pi r^2$ and its thickness ct

$\therefore$ Energy $W_E = \dfrac{q^2 a^2}{48\pi^2 \epsilon_0 c^4 r^2} \times 4\pi r^2 ct$

$= \dfrac{q^2 a^2 t}{12\pi \epsilon_0 c^3}$

See total energy is not depends on r. It means shell carries energy at fixed amount.

★ The amount of Energy lost by particles → stores to E field and it is fixed with Increased sphere.

Now for total field (as $\eta = \eta_E + \eta_B$)

$$W = W_E + W_B = \frac{q^2 a^2 t}{6\pi \epsilon_0 c^3}$$

$\therefore$ Power radiated $= \boxed{\dfrac{W}{t} = \dfrac{q^2 a^2}{6\pi \epsilon_0 c^3}}$ Larmor formula Provided $v \ll c$

$$\boxed{\text{Or} = \frac{2}{3} \frac{q^2 a^2}{4\pi \epsilon_0 c^3}}$$

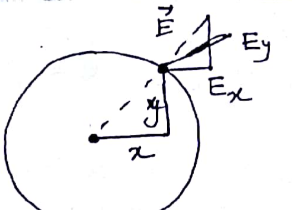
But what will happen if we consider it for Relativistic case ?? and also the effect on

$\therefore$ We know that there is a term involves $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\therefore E'_\perp = \dfrac{E_\perp}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$ and $E'_\parallel = E_\parallel$

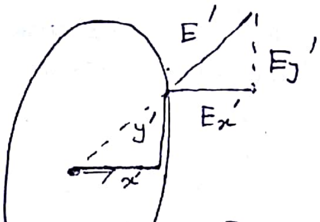
What will be the nature of E field when the charge is moving with uniform velocity. Let the charge is surrounded by a spherical shell and charge is rest at center. Now when charge and sphere both are moving then length contraction will happen and the sphere become spheroid.

Now consider the electric field at any point on the surface of sphere. if x and y are two components



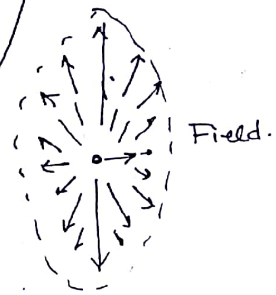
$$\frac{E_y}{E_x} = \frac{y}{x}$$

Now in reference frame x' has a length contraction by Lorentz factor. The contraction along charge motion. direction which is x



$$\therefore \quad x' = \sqrt{1-\left(\frac{v}{c}\right)^2}\; x$$

y remains same. But y component E field is $\perp^r$ to ~~prop~~ movement.



$$\text{So} \quad E_y' = \frac{E_y}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\therefore \quad \frac{E_y'}{E_x'} = \frac{E_y}{E_x\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{y}{x\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{y'}{x'}$$

it means that field in primed frame points directly away from charge as the original case. But faster the charge moving more the $E_\perp$ component enhances. If $v \ll c$ then this enhancement is negligible

What is length contraction :- It is a phenomenon of a decreases in length of an object as measured by an observer who is travelling at any non zero velocity relative to the object. It is noticable at a substantial fraction of speed of light. Length contraction is only along $\parallel^r$ direction.

$$L' = \frac{L}{\nu}$$

Now if we apply the same ν in accelerated particle then

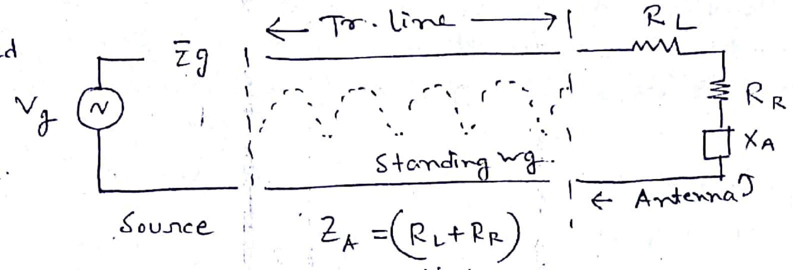
$\boxed{\eta \text{ will be a function of } \nu.}$

So when the charge has higher energy or higher mass we can not neglect ν and then length contraction at observation plane will come to play.

Antenna : Usually metallic Device for radiating or receiving radio waves.

Source is represented by ideal generator V_g & Z_g

Transmission line ch. impedance Z_L



Antenna is represented by Load Z_A.

$$Z_A = (R_L + R_R) + jX_A$$

R_L = Represents the conduction and dielectric loss associated with antenna structure.

R_r = Radiation resistance. (Radiation property of antenna)

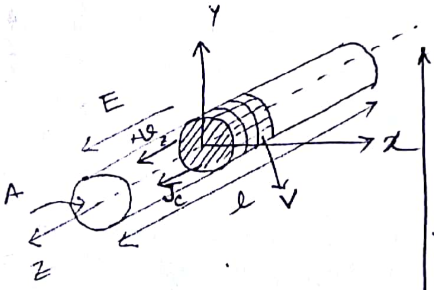
X_A = Imaginary part of the impedance associated with Radiation by antenna.

* Energy generated by the source should totally transferred to radiation resistance R_r. But the presence of conductor loss and dielectric loss exhibits loss in transmission line. Hence the total power may not fully transferred to antenna. Maximum power is delivered to the antenna under conjugate matching.
* Transmission line often called as feed line has to be designed properly. Other than that less amount of power will be transferred and power will be stored in a large amount inside transmission line. If the Input power is Large then at the peak of standing wave arching will happen inside tr. line.

Antennas

- Wire antenna
- Aperture antenna
- Frequency Independent antenna
- Micro strip antenna
- Leaky wave antenna
- Dielectric Resonator antenna
- Substrate integrated antenna
- Reflector antenna
- Lens antenna

Antenna Radiation Equation



Conducting wires are material whose prominent characteristic is the motion of charge and creation of current flow. Let us assume that an electric volume charge density of q_v C/m^3 is distributed uniformly in a circular wire of cross sectional area A and volume V. Total charge Q within the volume is moving in Z direction with uniform velocity v_z (mt/sec). Hence current density $J_z\,(Amp/m^2) = q_v \times v_z$ over the cross section.

Now if the wire is made of an ideal electric conductor then the current density J_s (amp/m) resides on the surface of the wire and is given by $q_s \times v_z$ where q_s (C/m^2) is the surface charge density.

Now if we consider that the wire is very thin (idealy zero radius) then the current in the wire depends on line charge density

$I_z = q_l v_z$ where q_l is ~~Colomb~~ coulombs/m [charge/length]

Now if this current is time varying then I_z can be written as dI_z/dt.

$$\Rightarrow \frac{dI_z}{dt} = q_l \frac{dv_z}{dt} = q_l \times a_z. \quad \text{where } a_z = mt/sec^2 \text{ is acceleration}$$

For total length l

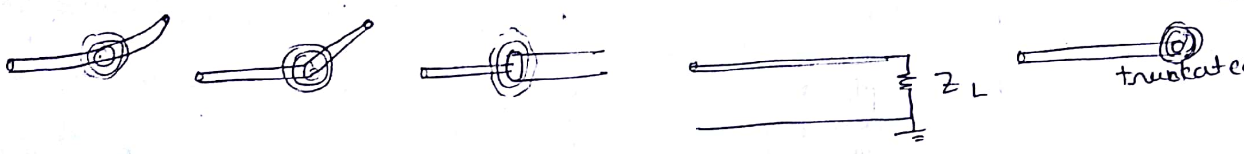
$$l\frac{dI_z}{dt} = l q_l a_z$$

This relation shows that to create radiation there must be a time varying current as time varying accelerated charge particles create radiation.

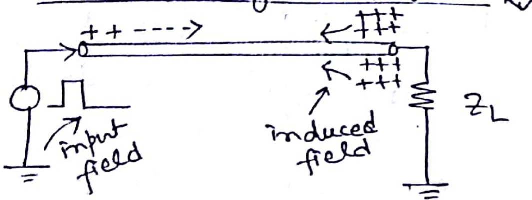
* Time variation of current can be pulsed or sinusoidal. For a pulsed input of current variation the radiation is for broader spectrum. For pure smooth sinusoidal variation the charge or current results a narrow Bandwidth of radiation.

To create charge acceleration, the wire must be curved, bent, discontinious. or terminated. Periodic charge acceleration or time varying current is also created when charge is ~~as~~ oscillating in time harmonic motion for a $\lambda/2$ length conductor. ②

Hence : if charge is not moving → no radiation
charge is moving with uniform velocity → no radiation if conductor is straight.
" " " " " " → if wire is bend then it radiates.
charge is oscillating in time motion → it radiates if wire is straight

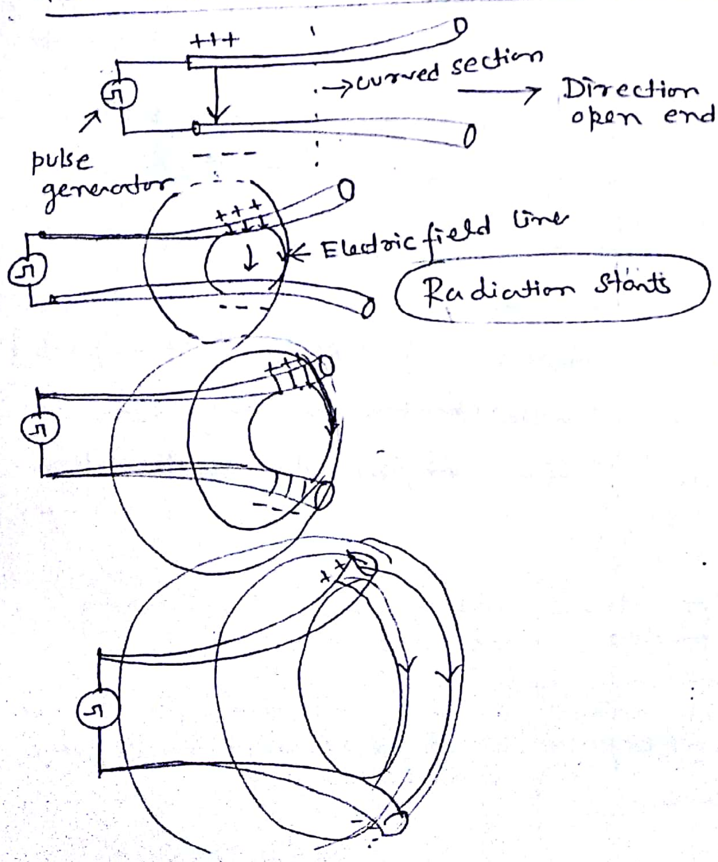


Understanding the physics (single wire)



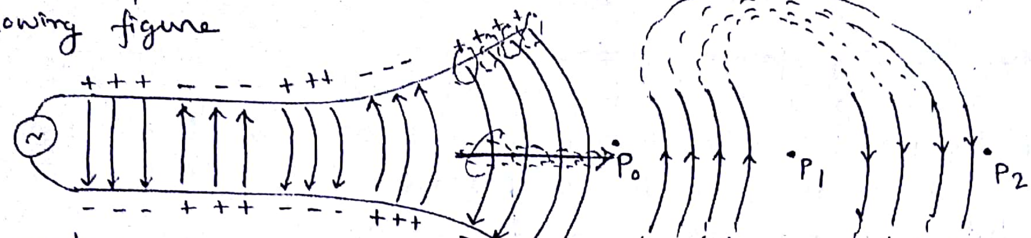
A pulse source is connected to an open ended conducting wire which may be connected to the ground by a load at that open end. When the ~~eh~~ wire is initially energized the free electrons in the wire are set in motion by the electric lines of force created by source. The acceleration of charge is accomplished by the external source in which force ~~is~~ set the charge in motion and produce acceleration. Hence in the source end there will be radiation associated to input field. Now charge build up happens at the end of the wire. Due to this the deceleration of the charge at open end is accomplished by the internal force associated with induced field. Hence charge will be declerated during the reflection from its end. As a result a radiated field will be generated. Internal force receive energy from the charge build up and it's velocity is zero at the wire end. As a result the whole wire contains acceleration and decleration of charge of different amount causing radiation.

How Radiation happens in a two wire transmission line

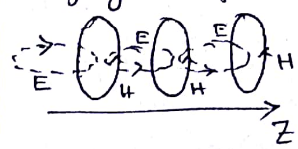


A single short pulse is used as excitation source which forces electric charge moving to the right along the uniform tr. line. The velocity is at speed of light. There is no radiation as the charge travels along the uniform section at the left side.
Now when charge reaches to the curved region then it undergo acceleration and radiation occurs. More and more the line bends acceleration of charge increases. Thus more radiation occurs. After the charge reaches at the end of the conductor it lost its amount of energy and hence negligible radiation occur on reflection of charge from open end.
With radiation from curved section, the energy of pulse decreases as energy is lost to radiation. That is why we can assume that due to prior radiation negligible charge reaches at the open end.

The same problem can be viewed in a discrete oscillating way by the following figure



A time varying E field produce a time varying H field



Two conductor transmission line creates electric field between conductor. Electric field is proportional to the electric field intensity. Movement of charge due to electric lines of force creates a current that in turn creates a magnetic field intensity. Electric lines of force starts with positive charge and end at negative charge. This phenomenon persists inside the two wire transmission line.

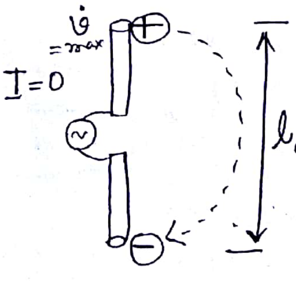
Now variation of electric field periodicity depends on sinusoidal oscillating source periodicity. Creation of time varying E and H field between conductor forms electromagnetic wave which travels along the transmission line.

Now at the open end free space waves can be formed by connecting the open ends of the electric lines. The waves are also periodic in nature and that is why they move with a constant phase point of. P_0 with a speed of light. In general $P_1 \rightarrow P_0$ is $\lambda/2$. close the antenna the constant phase point P_0 moves faster than the speed of light as it is still inside tr line but gradually it approaches to the speed of light at points far away from the antenna.

* Now it means that if you measure any pattern just outside the antenna you will get a wrong value. That is why the concept of nearfield and far-field measurement comes into this account.

Why the field lines outside the tr line is combined to form lobe

Consider a two wire line feed by a sinusoidal source as shown in the fig. We know a charge moving back and forth along the conductor is subject to acceleration and radiates.

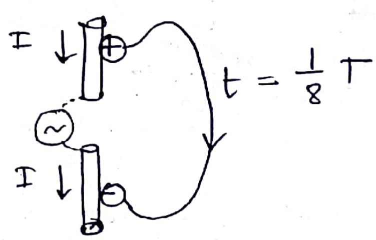


$t=0$ Two equal charge of opposite sign oscillating up and Down in harmonic motion.

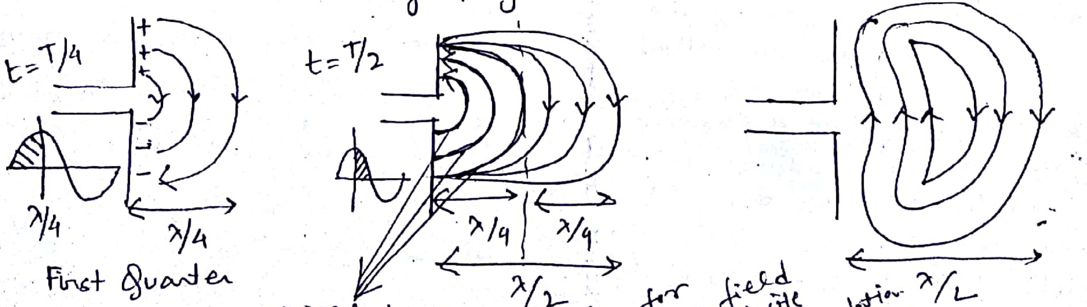
At t=0 charges are max separation and undergo max $\dot{v}$ at this instant $I=0$.

☆ Mind this point that no current in the rod and max field lines

$\Rightarrow$



Two Questions $\longrightarrow$ How they detached
Why they combined.

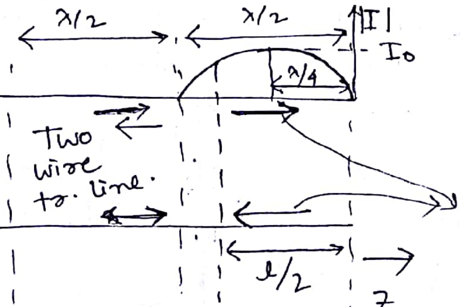


These are not actual field → These are the Representation for diminishing field due to opposite charge accumulation

First figures shows the lines of force created between the arms of small center fed dipoles excited by a sinusoidal signal and we are considering the first quarter of the period during which signal reaches it's peak value. Hence charge has reached its maximum value and hence lines have travelled max outward radial distance of $\lambda/4$. During next quarter of period the original lines traveled an additional $\lambda/4$ distance (total $\lambda/2$ from dipole). Now in this next $\lambda/2$, instance the signal strength diminishes. It means the charge strength starts reduce. This can be thought of as being accomplished by introducing opposite charge accumulation and ultimately at the end of first half $\lambda/2$ the charges are neutralized on conductor. The lines of force for this second $\lambda/4$ is of opposite direction.

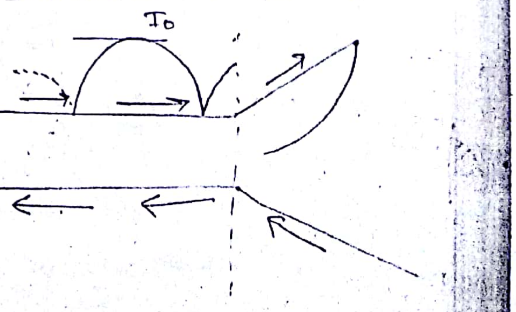
Now at this instant (at $\lambda/2$) no net charge on conductor or antenna. Hence no current at that instant. So no lines of force can be possible to be connected with conductor. It must be detached from conductor. As the whole incident happens instantaneously at $\lambda/2$ hence the prev $\lambda/4$ field lines and current $\lambda/4$ field lines can have instant connecting point at the edge of ~~line~~ antenna. Thus they form close loop [But why they are not connected in a point but produce a continuous loop. The reason is at the edge of the antenna there is singularity. Hence ~~to~~ there will be no way to calculate ~~sudden~~ electric field continuity in abrupt junction. Hence contour integral is required to make the loop continuous for a wide region.]

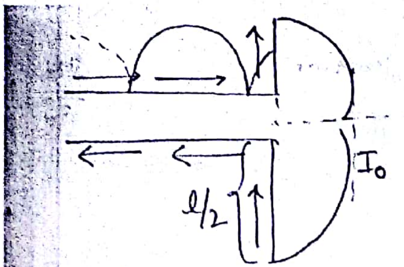
Current Distribution on a thin wire antenna — Concept of Dipole from two wire



This section exhibits the creation of current distribution on a twin wire line. This is the initial concept of Dipole. Movement of charge creates travelling wave current of magnitude $I_0/2$ along each wire. Direction of current in two wire is opposite. When current arrives at the end of each wire, it undergoes a reflection and thus standing wave generates with equal magnitude 180° phase reversal current component due to reflection. Radiation from each wire individually occurs as due to time varying nature of current. Note null at the end of line for current produce maximum of voltage and similarly max of current at $\lambda/4$ produce null of voltage. The current distribution in one wire is 180° out of phase from that in the corresponding half cycle of the other wire. More over if s is $\ll \lambda$ then field radiated by one wire are cancelled by the other wire. Hence it become non radiating.

Now in the next figure let a section of transmission line between $0 \leq z \leq l/2$ begins to flare. Current Distribution remains unaltered. Now as S is not too close hence fields are not cancelled at the flared region. Therefore ideally there is a net radiation by this system





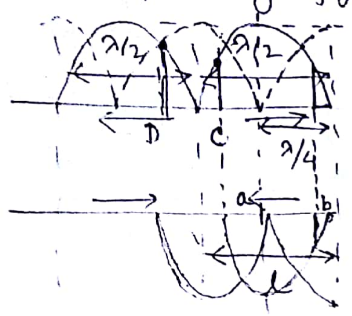
(E)

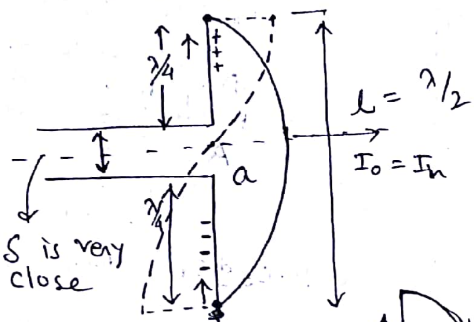
Now for maximum flane condition this geometry is called Dipole due to existance of two different charge at the extreme ends for maximum Radiation

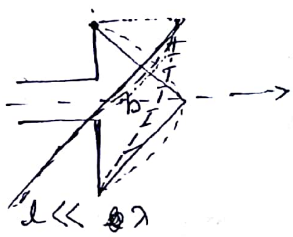
1. The current Direction in two arm are same having same Direction. It means the phone of current in standing wave are in phone.
2. Fields radiated from this two same orientation reinforce each other towand the direction of observation
3. The distance between two wire is important as depending upon current distribution zero field or null can be achieved at the center between two arm.

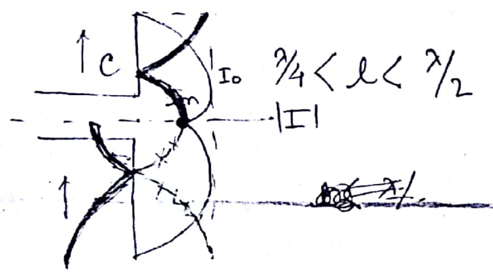
See the following figure to understand different length

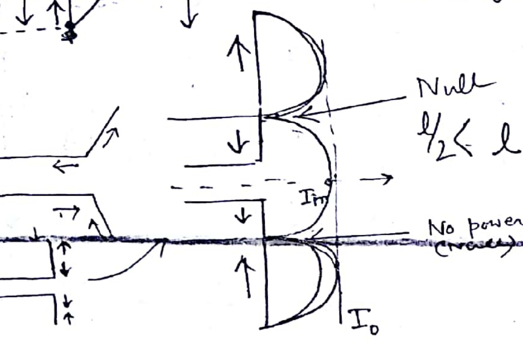
— Current Distribution
— Voltage Distribution











Some part are not in phone with other. Hence significant concelling of power can happen.

How to visualize the plane wave radiation from antenna.

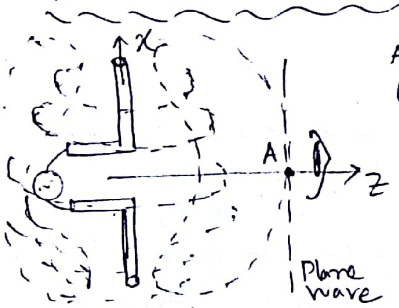


Fig 1

At the plane wave observation point obtained from electromagnetic wave it is evident that electric field is $\perp^r$ to the Direction of propagation along z and $\|^r$ to the expect antenna plane. That is why as per the definition the E plane is that plane in which Direction of which generates by the field lines of E field and direction of max radiation. So here xz phone is the E plane. Now at point A we can draw the field behavior as like Fig 2. If we imagine a sheet at A then E field and H(B) field can be visualized and they are travelling with velocity of c. Now question is that Is the field coming out from radiation follows Maxwell's equations ?

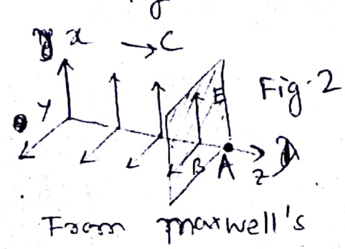


Fig 2

From maxwell's equation we know four laws

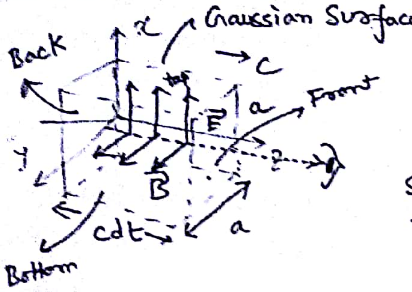
(A) Gauss's law $\oint \vec{E}\cdot dA = \frac{q_{in}}{\epsilon_0}$

(B) $\oint \vec{B}\cdot d\vec{A} = 0$

(C) $\oint B\cdot d\vec{l} = \mu_0 I_{enclosed} + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$

(D) $\oint E\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$

Proof of This 4 Laws on this observation point of A

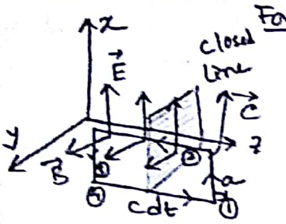


$\oint \vec{E}\cdot dA = Q_{in}/\epsilon_0 \Rightarrow$ Surface → how many

Front	Back	Side left	Side Right	top	Bottom		
0 +	0 +	0	+ 0	+ $\underbrace{Ea\,cdt}_{\text{Area of top}}$	$- Ea\,cdt$	$= 0 =$	$\frac{Q_{in}}{\epsilon_0}$

So it is proved that left side 0 and right side no enclosed charge.

Front	Back	Side Left	Side Right	top	Bottom	
0 +	0 +	$Bacdt$ +	$(-Ba\,cdt)$ +	0 +	0	= 0



Faraday's law $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$ $d\Phi_B \rightarrow$ change in magnetic flux. and $\vec{E}\cdot d\vec{l} = E\,dl\cos\theta$ For $\theta = 90°$ $\vec{E}\cdot d\vec{l} = 0$

① ② ③ ④

$0 + 0 + (-Ea) + 0$

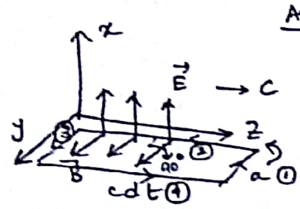
As wave has not reached

Direction of our motion is 180° out of phase wr Direction of E

Now if $dt \rightarrow 0$ then the length tends to zero which produce a thin sheet holding E. In this sheet also B remain constant as we are taking a very small portion so only A is variable. where A is the area

$\Rightarrow -Ea = -\frac{d\Phi_B}{dt}$

$\Rightarrow \cancel{-}E\cancel{a} = -B\frac{dA}{dt} = -B\frac{dacdt}{dt} = \cancel{-}B\cancel{a}c\frac{dt}{dt} \Rightarrow E = Bc$



Amp Law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{d\Phi_E}{dt}$

Now as no current for charge so $\mu_0 I_{enc} = 0$

Again $\vec{B}\cdot d\vec{l} = 0$ for $\theta = 90°$

① ② ③ ④

$0 + 0 + Ba + 0 = \mu_0\epsilon_0\frac{d\Phi_E}{dt}$

no wave

Now again if $dt \rightarrow 0$ the whole length reduced to a single filament.

So $Ba = \mu_0\epsilon_0\frac{d\Phi_E}{dt}$ $\Phi_E = E \cdot A$ Electric flux ⇒ Electric field times area.

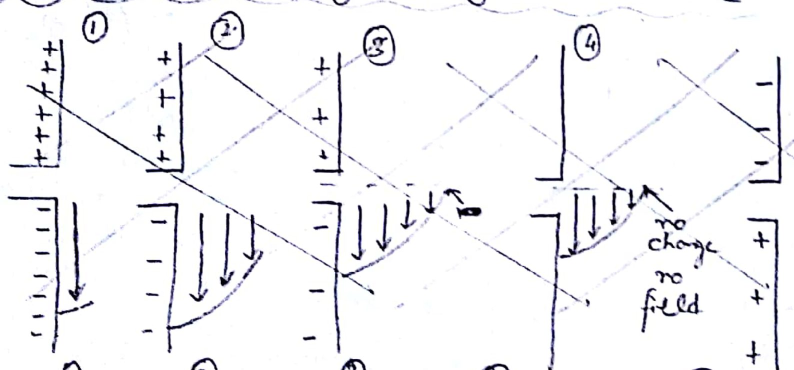
$B\cancel{a} = \mu_0\epsilon_0 E\frac{dA}{dt} = \mu_0\epsilon_0 E\cancel{a}c\frac{dt}{dt} = \mu_0\epsilon_0 Ec$

Now from Faraday's law $E = Bc$ → Putting it $B = \mu_0\epsilon_0 B\cdot c^2 \Rightarrow c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

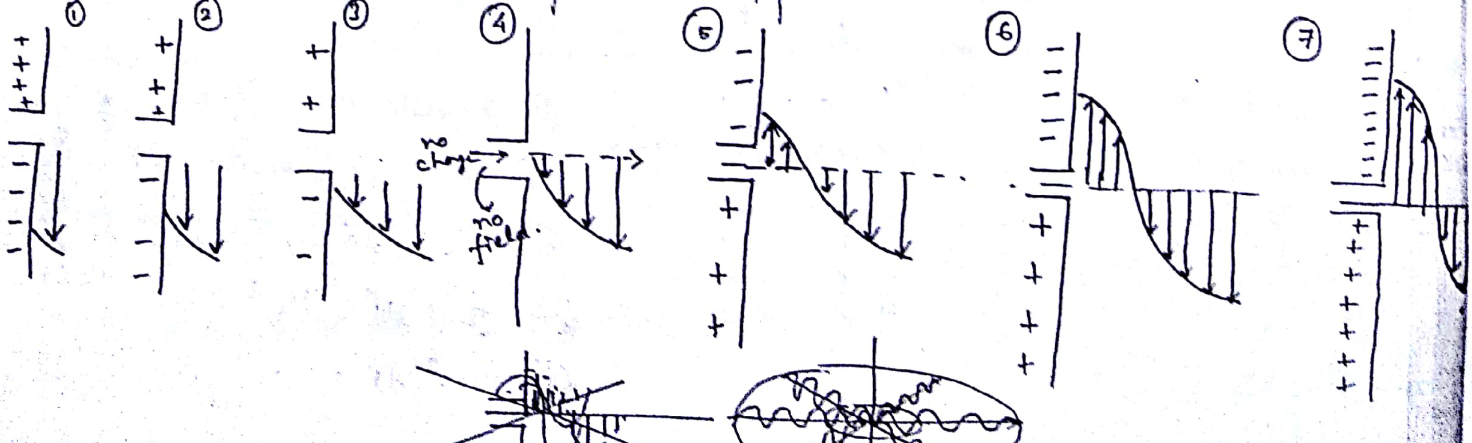
Now if you put μ_0 & ϵ_0 you will get $c = 3\times 10^8\,m/sec$

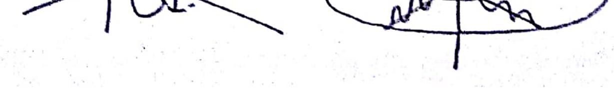
So it proves that the signal coming out with velocity of light follow Maxwell's eqn

Electric field coming out of twin wire



Electric field lines are coming out ⊥r to the direction of propagation. It produces the envelop of sine wave moving out in free space at a speed of light. This will continue with change in polarity





Proof of circular loop of magnetic field around a section of wire carrying current. (7)

How to get any magnetic field from current and go back from field to current

$$I_0 \partial \vec{l} \xrightarrow{\text{change}} \vec{H}(\vec{r_m})$$

Bio Savart law

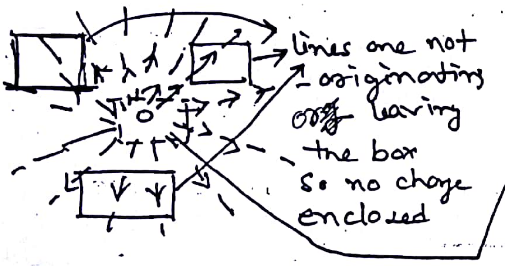
$$\vec{H}(\vec{r_m}) = \frac{I_0}{4\pi} \int \frac{\partial \vec{l} \times (\vec{r_m} - \vec{r_1})}{|\vec{r_m} - \vec{r_1}|^3}$$

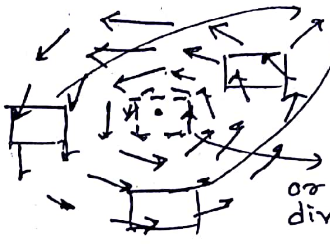
The problem is ~~cross~~ cross product there is a ~~cross~~ to be solved in the numerator to get H from I.
The process involved integration.

so is there any relation between B to I (Back method)

A) Gauss's law of magnetic field.

For electric field case $\nabla \cdot \vec{D} = \rho_V$ which means that if I have a box and if ~~charge~~ lines (electric fields) are entering (sink) or leaving (source) then only there will be a relation between flux vector to charge. Other than that it is zero.

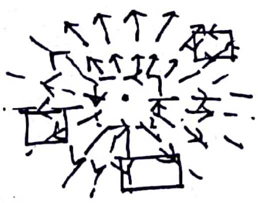




so $\nabla \cdot B = 0$

At no place there is any accumulation

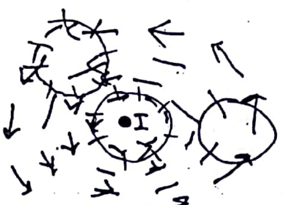
B) Ampere's circuital law



Around the length piece by piece integral will always provide zero and it remains at the center

$$\oint \vec{E}(\vec{r}) \cdot \partial \vec{l}(\vec{r}) = 0$$

Here every path is $\parallel^{el}$ to the swirling field. So piece by piece Integral will sums up to the current enclosed



$$\oint \vec{H}(\vec{r}) \cdot \partial \vec{l}(\vec{r}) = I$$

<u>Now</u>

$$I_0 \partial \vec{l} \xrightarrow{\text{Bio Savart Law}} \vec{H}(\vec{r_m})$$

$$\xleftarrow[\nabla \cdot \vec{B} = 0]{} \vec{B}(\vec{r_m})$$

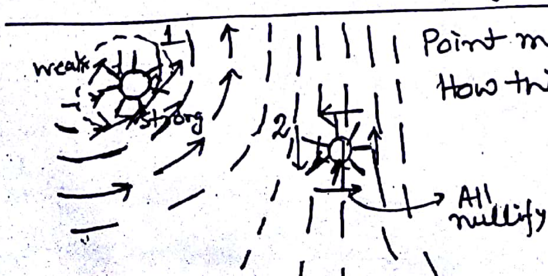
$$\oint \vec{H} \cdot (\vec{r}) \cdot \partial \vec{l}(\vec{r}) = I$$

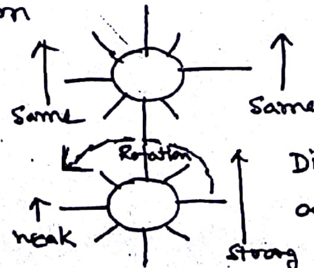
★ See the Diff. For Forward path there is integration and for backward path there is integration but in closed loop.

Secondly it tells you about the condition of a loop but for a space it is not stating anything.

So we need to find out a Differential form of the equation

Point measure of circulation
How this wheel rotate.

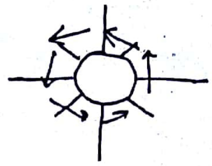




→ no Diff no rotation

Diff so rotation anticlockwise.

So in summary the Force exerted on the wheel around the loop of line can produce a circulation.

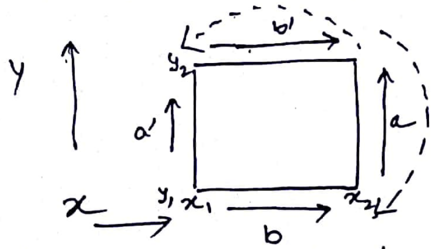


$$\oint \vec{F}(\vec{r}) \cdot \partial \vec{l}(\vec{r}) = \text{circulation.}$$

We can chang this integral operator to $\underline{\text{curl}}$ to have a Differential operation.

$$\oint F(\vec{r}) \cdot \partial \vec{l}(\vec{r}) = \text{circulation} \longrightarrow$$

Now it means we need to calculated the difference of force in each Diametrical opposite point and then sumps up the small discrete values. (see the concept)



we can compute a and a' and a − a' will inform me in which way I will rotate

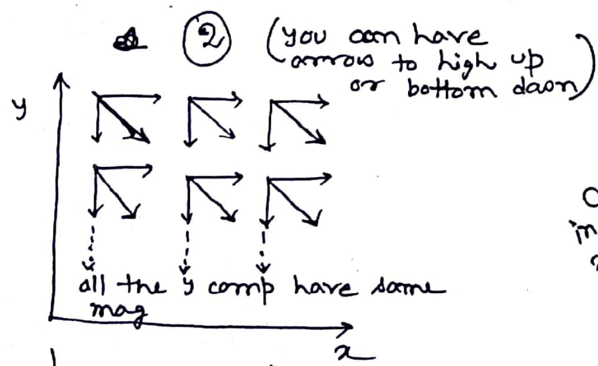
Similarly calculate b − b' and then find out among a − a' & b − b' which one Longer → that way it will rotate.

Now $a - a' \Rightarrow \Delta y$ $b - b' \Rightarrow \Delta x$ And again Δy for different instance of x $(x_1 \& x_2)$. Similarly Δx for different instant of y $(y_1 \& y_2)$.

So the $\boxed{\text{circulation}}$ can be think of in a other way

$$\left(\frac{d}{dx} F_y(x, y, z) - \frac{d}{dy} F_x(x, y, z) \right) \hat{z}$$

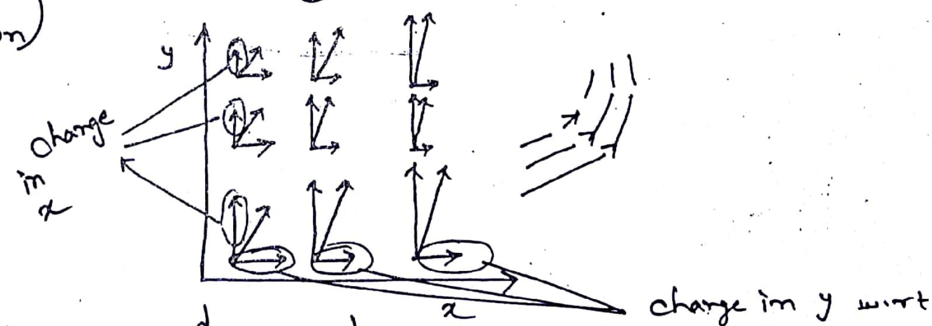
So ~~whit~~ with this now we can compute the cond^n of wheel



$$\frac{d}{dx} F_y - \frac{d}{dy} F_x$$

So no change in y comp w.r.t x

So $\frac{d}{dx} F_y - \frac{d}{dy} F_x = 0$



$$\frac{d}{dx} F_y - \frac{d}{dy} F_x$$

$$(+) - (-)$$

$(+)$ So there is some rotation or circulation.

Now we can visualize in a general way

$$\vec{\nabla} = \frac{d}{dx} \hat{x} + \frac{d}{dy} \hat{y} + \frac{d}{dz} \hat{z}$$

variation of x in x direction
variation of y in y direction

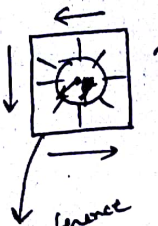
$$\nabla \times \vec{F} = \left(\frac{d}{dy} \vec{F_z} - \frac{d}{dz} \vec{F_y} \right) \hat{x} + \left(\frac{d}{dz} \vec{F_x} - \frac{d}{dx} \vec{F_z} \right) \hat{y} + \left(\frac{d}{dx} \vec{F_y} - \frac{d}{dy} \vec{F_x} \right) \hat{z}$$

in our above calculation only third term will exists.

So $\oint \vec{F}(\vec{r}) \cdot \partial \vec{l}(\vec{r}) \Leftrightarrow \vec{\nabla} \times \vec{F}$

Now to convert integral to Differential operator we got the famous law Stokes' Law

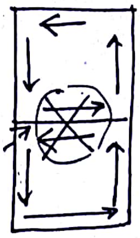
Stokes law



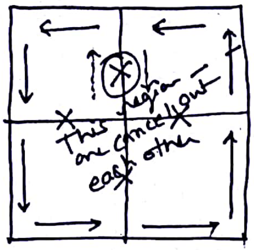
circumference

$$\oint \vec{F}(\vec{r}) \cdot \partial\vec{l}(\vec{r}) = \nabla \times \vec{F}$$

circulation can be measured either by sum of line integration of forces around the path or by curl of force.



$$\oint \vec{F}(\vec{r}) \cdot \partial\vec{l}(\vec{r}) = \underset{①}{\vec{\nabla} \times \vec{F}} + \underset{②}{\vec{\nabla} \times \vec{F}}$$



- So the integral remains on surface

$$\oint \vec{F}(\vec{r}) \cdot \partial\vec{l}(\vec{r}) = \iint (\vec{\nabla} \times \vec{F})\, \partial s$$

If we sumps up the integration over the surface around the structure the same thing will come either by moving around a line or moving on a surface

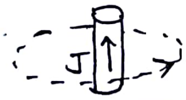
Now we replace F by H as H is the field which can exerts force for circulation.

$$\oint \vec{H}(\vec{r})\, \partial l(\vec{r}) = I = \iint (\nabla \times \vec{H})\, \partial s$$

$$I = \iint J \cdot ds$$ | For a surface current density J over a surface ds

$$\therefore \iint J \cdot ds = \iint (\nabla \times \vec{H})\, \partial s$$ | both side surface integral.

$$\Rightarrow \boxed{J = \nabla \times H}$$



$J = \nabla \times H$

So if there is no $\vec{J}(=0)$ or a source free region $\nabla \times H = 0$

by $\boxed{\text{Irrotational theorem}}$ $\vec{H} = -\nabla V_m$ [curl less field]

From vector irrotational law if $\vec{F}$ is vector then $\nabla \times \vec{F} = 0$ if

$\int_a^b \vec{F} \cdot dL$ = independent of path & $\oint \vec{F} \cdot dL = 0$

$\vec{F} = -\nabla V_m$ where V_m is a scalar quantity known as $\boxed{\text{Scalar potential}}$

Adv: easier to handle a scalar quantity over vector quantity. Applicable only in source free region.

So from $\vec{H} = -\nabla V_m$ we can say $\vec{H}$ = magnetic field intensity and V_m is a scalar quantity related to magnetic field intensity. So V_m is called magnetic Scalar potential.

Secondly $\nabla \cdot \vec{B} = 0$ and $\vec{B} = \mu\vec{H}$ $\Rightarrow -\mu \nabla \cdot \nabla V_m = 0 \Rightarrow \nabla^2 V_m = 0$

Thus magnetic scalar potential satisfies the laplace's equation.

Now again by $\boxed{\text{Solinoidal theorem}}$ or $\boxed{\text{Divergence Less Field}}$

$\nabla \cdot \vec{F} = 0 \Rightarrow \int_{surf} \vec{F} \cdot d\vec{a}$ = indp of surface $\Rightarrow \oint_{surf} \vec{F} \cdot d\vec{a} = 0 \Rightarrow \vec{F} = \nabla \times \vec{W}$

$\boxed{\text{Vector potential}}$

From this $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$ As B is magnetic so it is magnetic Vector Potential

Now from Ampere's law $\nabla \times \vec{H} = \vec{J}$ [Here no source free Region]

So $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} \Rightarrow \frac{1}{\mu} \vec{\nabla} \times \nabla \times \vec{A} = J \Rightarrow \boxed{\nabla \times \nabla \times \vec{A} = \mu\vec{J}}$

$$\Rightarrow \nabla \underbrace{\nabla \cdot \vec{A}}_{0} - \nabla^2 \vec{A} = \mu J$$

So $\nabla^2 \vec{A} = -\mu \vec{J} \Rightarrow$ Vector Poisson's Equation

vector operator operated

So $A = \int_L \mu \frac{J_L \, d\vec{l}'}{4\pi R}$ R is $(r - r')$

line current

$A = \iint_S \mu \frac{K}{4\pi R} d\vec{s}$ $A = \iiint_V \frac{\mu J_V}{4\pi R} d\vec{V}$

What is the Relatioship of Magnetic Vector Potential (A) and Electric Currer Source J in terms of electric Scalar potential.

Vector Potential can be defined as auxiliary function.

A is generated due to harmonic electric current J. The magnetic field is always solinoidal as we have proved that $\nabla \cdot B = 0$. Therefore we can say that

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Again $\vec{B} = \mu \vec{H} = \nabla \times \vec{A}$ So $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$

Now From the Maxwell's curl equation we can write that

$\nabla \times E = -\frac{\partial B}{\partial t}$ | $\nabla \times \vec{E}_A = -j\omega\mu \vec{H}_A = -j\omega\mu \frac{1}{\mu}(\nabla \times \vec{A}) = -j\omega(\nabla \times \vec{A})$

Thus $\nabla \times (\vec{E}_A + j\omega\vec{A}) = 0$

Again From vector identity we know $\nabla \times (-\nabla \phi_e) = 0$ | From irrotational theorem.

$$\vec{E}_A + j\omega\vec{A} = -\nabla\phi_e$$

$$\Rightarrow \boxed{\vec{E}_A = -\nabla\phi_e - j\omega\vec{A}}$$

Here ϕ_e represents an arbitrary electric scalar poten which is function of position. Similar to prev page curl less field E field is irrotational and hence $\nabla \times E = 0$. Related to E field hence ϕ_e is electric scal potentia

Now taking the identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A$$

$$\Rightarrow \nabla \times (\mu H_A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

For a homogeneous medium $\mu(\nabla \times H) = \nabla(\nabla \cdot A) - \nabla^2 A$

Again we know from Maxwell's equation $\nabla \times H = J + j\omega\epsilon E_A$

So Putting this value in prev equation

$$\mu J + j\omega\epsilon\mu E_A = \nabla(\nabla \cdot A) - \nabla^2 A$$

Replacing $E_A = -\nabla\phi_e - j\omega A$ $\Rightarrow$ $\mu J + j\omega\epsilon\mu(-\nabla\phi_e - j\omega A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$\Rightarrow -j\omega\epsilon\mu\nabla\phi_e - j^2\omega^2\epsilon\mu A + \mu J = \nabla(\nabla \cdot A) - \nabla^2 A \Rightarrow -j\omega\epsilon\mu\nabla\phi_e + \omega^2\epsilon\mu A + \mu J = \nabla(\nabla \cdot A) - \nabla^2 A$

$$\nabla^2 A + \omega^2\mu\epsilon A = -\mu J + \nabla(\nabla \cdot A) + j\omega\epsilon\mu\nabla\phi_e = -\mu J + \nabla(\nabla \cdot A + j\omega\mu\epsilon\phi_e)$$

and $K^2 = \omega^2\mu\epsilon$ propagation term

Now if we take $\boxed{\nabla \cdot A = -j\omega\mu\epsilon\phi_e \Rightarrow \phi_e = -\frac{1}{j\omega\epsilon\mu}\nabla \cdot A}$ ← Condition Lorentz Gauge *

Proves to the Lorentz condition of $\boxed{\nabla^2 A + k^2 A = -\mu J}$ Final equation.

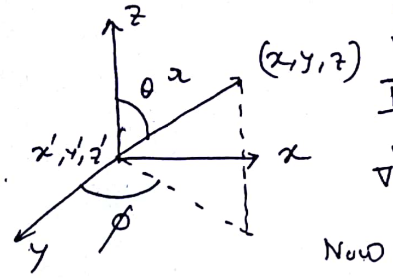
Then $E_A = -\nabla\phi_e - j\omega A = -j\omega A - j\frac{1}{\omega\epsilon\mu}\nabla(\nabla \cdot A)$

So if you know $A \to H_A$ can be obtained from $\boxed{H = \frac{1}{\mu}(\nabla \times A)}$ and

$$\boxed{E_A = -j\omega A - j\frac{1}{\omega\epsilon\mu}\nabla(\nabla \cdot A)}$$

$* \Rightarrow$ in general $\boxed{\nabla \cdot A = -\mu_0\epsilon_0 \frac{\partial\phi_e}{\partial t}}$

Now in the previous section we got the effect of A. Now we need to derive the A which is the solution of inhomogeneous vector wave equation.
Let us take a picture where at the center J is the current density and so it is z directed so we can assume J_z. It is an infinitesimal source. Only A_z component from A will exist as J_z is present.
So from prev equation we can write $\nabla^2 A_z + k^2 A_z = -\mu J_z$. Now if we reduce the height of current source so that the source become a point where $J_z = 0 \Rightarrow \nabla^2 A_z + k^2 A_z = 0$
Now no as A_z is related to a point source so we can assume A_z is not a f^n of θ & ϕ for spherical coordinates. $A_z = A_z(r)$ where r is the radial distance



Now we need to solve $\nabla^2 A_z$ as like solution of Poissons equation.

In general

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 F}{\partial\phi^2}$$

Now here F is replaced by A & $A(r)$. $\underset{=0}{A(\theta)}$ & $A(\phi) = 0$

So $$\nabla^2 A_z(r) + k^2 A_z(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z(r)}{\partial r}\right) + k^2 A_z(r) = 0$$

Expanding. $$\frac{d^2 A_z(r)}{dr^2} + \frac{2r}{r^2}\frac{dA_z(r)}{dr} + k^2 A_z(r) = 0$$

$$\Rightarrow \frac{d^2 A_z(r)}{dr^2} + \frac{2}{r}\frac{dA_z(r)}{dr} + k^2 A_z(r) = 0$$

Now if we solve this then A will have two solution

$$A_{z1} = C_1 \frac{e^{-jkr}}{r} \quad \& \quad A_{z2} = C_2 \frac{e^{+jkr}}{r}$$

$e^{-jkr} \rightarrow$ outwardly travelling wave e^{jkr} is inwardly travelling wave
and for all case $e^{j\omega t}$ is the time variation.
Here source / point charge is placed at the center and wave is coming out in all direction. Therefore we choose only one solution.

$A_z = C_1 \frac{e^{-jkr}}{r}$ Now for static case $\omega = 0$ $k = 0$ hence $A_z = \frac{C_1}{r}$

So when the point of calculation is removed from source, and it is time varying then the o/p will have a new extra component of e^{-jkr} with above C_1/r

Now when $J_z \neq 0$ then R.H.S $\neq 0$ but if still $k = 0$

$\boxed{\nabla^2 A_z = -\mu J_z}$ This is simple Poissons equation

Now comparing with Potential ⟺ Electric field ⟺ Charge we can find sol^n of Poisson equation. For this Coulombs law is also necessary to know

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\epsilon}\frac{q}{r}$$

In general $$V(P) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho}{r}d\tau \approx \frac{1}{4\pi\epsilon_0}\iiint\frac{\rho}{r}dv'$$

So $$A_z = \frac{\mu}{4\pi}\iiint_V \frac{J_z}{r}dv'$$ [Duality] [k=0]

For time varying sol^n

$$A_z = \frac{\mu}{4\pi}\iiint_V J_z \frac{e^{-jkr}}{r}dv'$$

Now if J has J_x & J_y then sol^n will be divided as per axis

$$\nabla^2 A_x + K^2 A_x = -\mu J_x \quad \text{and} \quad \nabla^2 A_y + K^2 A_y = -\mu J_y$$

and hence $A_x = \frac{\mu}{4\pi} \iiint_V J_x \frac{e^{-jkr}}{r} dv'$ & $A_y = \frac{\mu}{4\pi} \iiint_V J_y \frac{e^{-jkr}}{r} dv'$

In general $\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J} \frac{e^{-jkr}}{r} dv'$

If it happens that source ~~x', y', z'~~ is removed from origin and placed at x', y, z' then

$$A(x, y, z) = \frac{\mu}{4\pi} \iiint_V \vec{J}(x', y', z') \frac{e^{-jkR}}{R} dv'$$

where $\vec{R}$ is vector diff between r' & r. r' & r are the distance from origin.

Now we need to find out E & H field as we got the information (A).

~~Now in spherical coordinates~~

$$A = \hat{a}_r A_r(r, \theta, \phi) + \hat{a}_\theta A_\theta(r, \theta, \phi) + \hat{a}_\phi A_\phi(r, \theta, \phi)$$

Now $A = \frac{\mu}{4\pi} \iiint_V J \frac{e^{-jkr}}{r} dv' \Rightarrow dv' \Rightarrow dr, d\theta, d\phi$

So $= \frac{\mu}{4\pi} \int \text{for } dr \int \text{for } d\theta \int \text{for } d\phi$

Now if we assum $\vec{J}$ has variation only on r so no variation for θ & $\phi \Rightarrow$ This is for simplicity

Now $\frac{d}{dr}\int \frac{e^{-jkr}}{r} dr$ always produce ~~later~~ $\frac{1}{r^n}$ if we expand e^{-jx} by Binomial expansion.

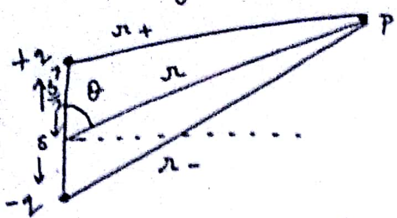
We know $e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ Required Explained

So $A \cong \left[\hat{a}_r A_r'(\theta, \phi) + \hat{a}_\theta A_\theta'(\theta, \phi) + \hat{a}_\phi A_\phi'(\theta, \phi)\right] \frac{e^{-jkr}}{r}$ $r \to \infty$ and

We are taking out r dependencies as in general $\frac{1}{r^n}$ terms are coming and if $r \to \infty$ then $\frac{1}{r^n} = 0$ for all $n = 2, 3 \cdots$

Now we know $E = -j\omega A - j\frac{1}{\omega\mu\epsilon} \nabla(\nabla \cdot A)$

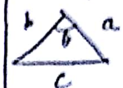
Consider the Figure where two opposite charge is placed on a line



The charge placed is an electric dipole which consists two equal and opposite charge ($\pm q$) separated by a distance s. What is the potential at P far away. Let r_- is the distance from $-q$ and r_+ is the distance from $+q$.

$$V(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

Law of cosine

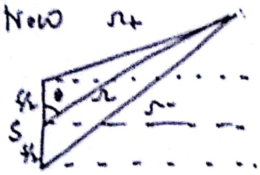


$c^2 = a^2 + b^2 - 2ab\cos C$

Imagine Pithagorus th. where $\theta = 90°$ then $c^2 = a^2 + b^2$

How $r_+^2 = r^2 + \left(\frac{s}{2}\right)^2 - 2 \cdot r \cdot \frac{s}{2}\cos\theta$

Now r_+



$\left(\frac{s}{2}\right)$ one arm

r one arm

$r\cos\theta$ } component of r
$r\sin\theta$

$$r_+^2 = r^2 + \left(\frac{s}{2}\right)^2 - rs\cos\theta$$

$$r_-^2 = r^2 + \left(\frac{s}{2}\right)^2 + rs\cos\theta$$

$$\therefore \quad r_\pm^2 = r^2\left(1 + \frac{s^2}{4r^2} \mp \frac{s}{r}\cos\theta\right)$$

we are calculating $r \gg s$ so $\frac{1}{r^2}$ is very small.

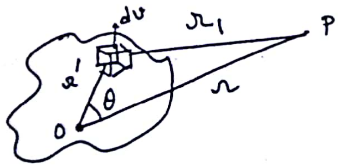
$$\therefore \quad \frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \mp \frac{s}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{s}{2r}\cos\theta\right)$$

$$\therefore \quad \left(\frac{1}{r_+} - \frac{1}{r_-}\right) = \cancel{\frac{1}{r}} + \frac{s}{2r^2}\cos\theta - \cancel{\frac{1}{r}} + \frac{s}{2r^2}\cos\theta = \frac{s}{r^2}\cos\theta$$

$$\therefore \quad V(P) = \frac{1}{4\pi\epsilon_0}\frac{q\,s\cos\theta}{r^2}$$

So for a dipole charge Potential content $\frac{1}{r^2}$. It falls of rapidly than the potential of a distribution with nonzero charge. Now for quadrapole the potential have $\frac{1}{r^3}$ for octopole $\frac{1}{r^4}$. For monopole $\frac{1}{r}$

So we need a general equation for the $\left(\frac{1}{r^n}\right)$



From this Figure we need to find out potential at P

$$V(P) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r_1}\rho\, dv$$

ρ = volume charge density at dv

r_1 is the distance from dv to P.

$$r_1^2 = r^2 + (r')^2 - 2rr'\cos\theta$$

$$= r^2\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right) \Rightarrow$$

~~and~~ $r_1 = r\sqrt{1+\epsilon}$ where $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$

Now if P is such that r is $\gg r'$ then ϵ is $\ll 1$

$$\therefore \quad \frac{1}{r_1} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left\{\left(\frac{r'}{r}\right)^2 + 4\cos^2\theta - 4\frac{r'}{r}\cos\theta\right\} - \cdots\right]$$

$$\Downarrow$$

$$+ \frac{3}{8}\left(\frac{r'}{r}\right)^4 + \left(\frac{r'}{r}\right)^2\frac{3}{2}\cos^2\theta - \frac{3}{2}\left(\frac{r'}{r}\right)^3\cos\theta$$

$$\frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)^2+\left(\frac{r'}{r}\cos\theta\right)+ \cancel{f}\left(\frac{r'}{r}\right)^2\left\{\frac{3}{8}\left(\frac{r'}{r}\right)^2+\frac{3}{2}\cos^2\theta-\frac{1}{2}\frac{r'}{r}\cos\theta\right\}---\right] \quad (15)$$

$$\approx \frac{1}{r}\left[1+\frac{r'}{r}\cos\theta+\left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)+\cdots\right]$$

we need to contentrate on power series of $\frac{r'}{r}$. This is called legendre Polynomial

$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$
$n=0 \quad P_n(x)=1$
$n=1 \quad P_n(x)=x$
$n=2 \quad P_n(x)=\frac{1}{2}(3x^2-1)$

$$\frac{1}{r_1}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

here $x=\cos\theta$

P_n are in the ( ) terms like
$P_n=1$ for First term
$P_n=\cos\theta$ second "
$P_n=\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$ 3rd term

$$\therefore V(P)=\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta)\,\rho\, dv$$

why I made r outside $\int$ but r' inside $\int$ as the obaervation point and o are fixed $\Rightarrow$ r is constant but r' is variable as we can choose any region to calculate.

$$V(P)=\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho\, dv+\frac{1}{r^2}\int r'\cos\theta\,\rho\, dv+\frac{1}{r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho\, dv\right]$$

This is the multipole expansion of V for $\frac{1}{r}$. The first term is for monopole. Second term is for dipole. The third term is for quadrapole. ----

So the most dominating term is for mono pole which is

$$V_{mono}(P)=\frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad Q=\int\rho\, dv \qquad Q \Rightarrow \text{at origin}$$

For Dipole total net charge is zero at origin

$$V_{dipole}(P)=\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta\,\rho\, dv$$

Now θ is the angle between r' and r $\Rightarrow$ $r'\cos\theta=\hat{r}\cdot r'$

$$\therefore V(P)=\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int r'\rho\, dv$$

→ Look at this integral. It does not depends on the location of P. This is called Dipole moment (P)

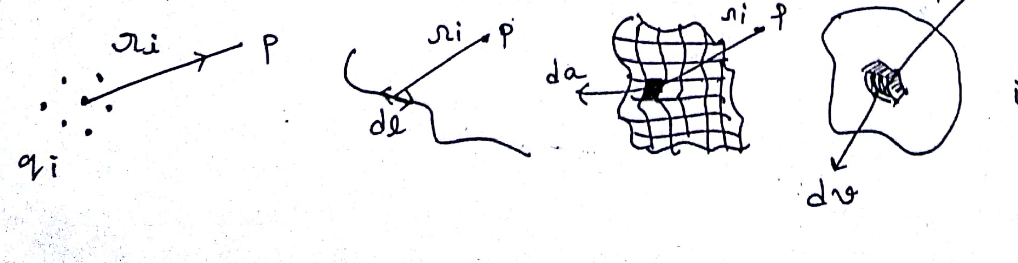
$$P=\int r'\rho\, dv$$

ρdv
length (r')

$$\therefore V_{dip}(P)=\frac{1}{4\pi\epsilon_0}\frac{P\cdot\hat{r}}{r^2}$$

Dipole moment is depending on geometry size, shape, density.

Now Three situation can arrise



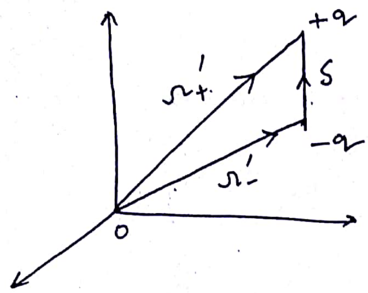
So q_i can be replaced by $dq=\lambda\, dl$ σda or ρdv

$$\sum_{i=1}^{n} q_i(r_i)\approx\int_{line}\lambda\, dl \approx\int_{sur}\sigma\, da \text{ area} \approx\int_{vol}\rho\, dv \text{ volume}$$

So $\Pi = \int r' \rho \, dv \Rightarrow \sum_{i=1}^{n} q_i r_i'$

For physical Dipole where only two charge present

$$\Pi = q r_+' - q r_-' = q(r_+' - r_-') = qs$$



Putting ~~P~~ = qs in general V_{dip} expression, we get

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{\Pi \cos\theta}{r^2} = \frac{qs \cos\theta}{4\pi\epsilon_0 r^2}$$

Now you see if we take all the component of r and if $r \not\gg r'$ then we can not neglect r. So as we are near to the dipol we can not get approximated voltage but more we go further away it becomes better and better as higher terms rapidly die off.

Concept of Retarded Potential.

We know from the concept of radiation that a charge in rest does not generate any radiation and accelerating charge produce radiation. But the question is when you are getting a radiation is it the effect of present condition or is it the previous condition?

We know $E = -\nabla V - \frac{\partial A}{\partial t}$ and $B = \nabla \times A$ so after combining $\nabla \cdot A = -\frac{1}{c^2}\frac{\partial V}{\partial t}$ interms of Helmhot

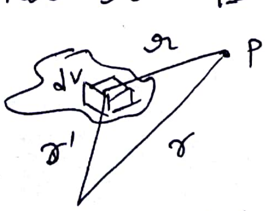
we can define the scalar potential V and vector potential A wave equation by

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon}\rho \quad \text{and} \quad \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

If the second term in both L.H.S = 0 ~~then~~ and $k = 0$ so then it will be static case. So this equations are inhomogeneous wave equation and for source free region it will be homogeneous wave equation.

For static case we know $V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{r} dv$ and $A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r')}{r} dl$

where $r =$ ~~$|r - r'|$~~ $|r - r'|$ is the distance from source to point P.



In nonstatic case it is not the condition of source right now that matters, but rather its condition at some earlier time t_r (retarded time) when the message left. It is required some time to travel a distance r to reach P. So the delay is r/c.

Now electromagnetic signal travels at a speed of light.

For simplicity and better understanding $r \equiv$ denoting by R as all looks similar.

Now if t_r is the time to travel then

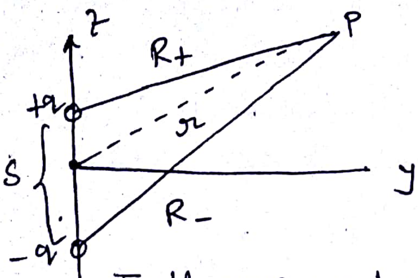
$$t_r = t - \frac{R}{c}$$

$$\therefore \; V(r,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r', t_r)}{R} dv \quad \text{and} \quad A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r', t_r)}{R} dv$$

charge $\rho(r', t_r)$ is the charge density that prevailed at point r' at t_r. This kind of potential measured at retarded time is called retarded potential. It is like our observation of stars, light we see is back time light from which lights are coming but what we observe is

We can not use any condition where $t_a = t + \frac{R}{c}$ as it is against causality.

Now using the concept of Retarded potential find out the Dipole Radiation and field.



Two tiny metal seperated by a distance s connected by thin wire. The whole system is electrically neutral. At time t the charge on the upper sphere is $q(t)$ and lower sphere is $-q(t)$. Charge is back and forth through the wire with a frequency ω as $q(t) = q_0 \cos \omega t$.

Till now we have measured what will be V for the same kind of picture and the effects of higher order terms. But that was for static condition means always $+q$ and $-q$ are unchangeable. Here they will oscillate.

Now as previous we know Dipole moment $\vec{p}(t) = q(t) \cdot s = q_0 s \cos \omega t \, \hat{k}$

so $p_0 = q_0 s$ is the maximum value of dipole moment under oscillation.

Now Retarded potential is given by

$$V(r, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0 \cos\omega (t - R_+/c)}{R_+} - \frac{q_0 \cos \omega\left(t - \frac{R_-}{c}\right)}{R_-}\right]$$

Again $R_+^2 = r^2 - rs\cos\theta + \left(\frac{s}{2}\right)^2$ and $R_-^2 = r^2 + rs\cos\theta + \left(\frac{s}{2}\right)^2$

Now for a perfect Dipole is $s \ll r$

$\cancel{R_\pm = r\left(1 \mp \frac{1}{2}\frac{s}{r}\cos\theta\right)}$ $\Rightarrow \frac{1}{R_\pm} = \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{s}{r}\cos\theta\right)$

~~Again~~ ~~cos ω~~ Putting the above condition in Cos part

$$\cos\omega\left(t - \frac{R_+}{c}\right) = \cos\left[\omega\left(t - \frac{\left(r - \frac{1}{2}s\cos\theta\right)}{c}\right)\right]$$

$$= \cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega s\cos\theta}{2c}\right]$$

Similarly $\cos\omega\left(t - \frac{R_-}{c}\right) = \cos\left[\omega\left(t - \frac{r}{c}\right) - \frac{\omega s}{2c}\cos\theta\right]$

Now expanding

$$\cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega s\cos\theta}{2c}\right] = \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega s\cos\theta}{2c}\right) - \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega s\cos\theta}{2c}\right)$$

$$\cos\left[\omega\left(t - \frac{r}{c}\right) - \frac{\omega s\cos\theta}{2c}\right] = \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega s\cos\theta}{2c}\right) + \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega s\cos\theta}{2c}\right)$$

We can now apply an approximation that as s is very small $s \ll \frac{c}{\omega}$

Because $\omega = 2\pi f$ so $\lambda = \frac{2\pi c}{\omega}$ $s \ll \frac{c}{2\pi f}$ $s \ll \frac{c}{\frac{2\pi c}{\lambda}}$ $s \ll \lambda$

Then $\cos\omega\left(t - \frac{R_\pm}{c}\right) = \cos\omega\left(t - \frac{r}{c}\right) \boxed{\text{Second Part tends to 1}} \mp \sin\omega\left(t - \frac{r}{c}\right)\left(\frac{\omega s}{2c}\cos\theta\right)$

Now

$$V(r, \theta, t) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega s}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)}{r\left(1 - \frac{1}{2}\frac{s}{r}\cos\theta\right)} - \frac{\cos\omega\left(t - \frac{r}{c}\right) + \frac{\omega s}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)}{r\left(1 + \frac{1}{2}\frac{s}{r}\cos\theta\right)}\right]$$

$$= \frac{q_0}{4\pi\epsilon_0}\left[\frac{\left(r + \frac{1}{2}s\cos\theta\right)\cos\omega\left(t - \frac{r}{c}\right) - \left(r + \frac{1}{2}s\cos\theta\right)\frac{\omega s}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right) - \left(r - \frac{1}{2}s\cos\theta\right)\cos\omega\left(t - \frac{r}{c}\right) - \cdots}{r^2 - \frac{1}{4}s^2\cos^2\theta}\right]$$

cross multiply and expand

In denominator take approx $r^2 - \frac{1}{4}s^2\cos^2\theta \approx r^2$

$$= \frac{p(q_0 s)\cos\theta}{4\pi\epsilon_0 r}\left[-\frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right)\right]$$

So we got

$$V(r,\theta,t) = \frac{P_0 \cos\theta}{4\pi\epsilon_0 r}\left[-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right]$$

Now for static case $\omega = 0$ so $\boxed{V = \frac{P_0 \cos\theta}{4\pi\epsilon_0 r^2}}$ which is same as prev expression

But here $\omega \neq 0$ and as we choose $S \ll \frac{c}{\omega}$ then $r \gg \frac{c}{\omega}$ or $r \gg \lambda$

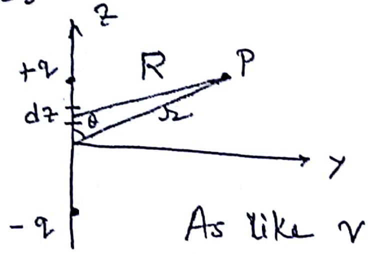
then

$$\boxed{V(r,\theta,t) = -\underbrace{\frac{P_0\omega}{4\pi\epsilon_0 c}}_{\text{Medium}}\underbrace{\left(\frac{\cos\theta}{r}\right)}_{\text{Axis}}\underbrace{\sin\omega\left(t-\frac{r}{c}\right)}_{\text{Osc freq and Retarded time}}}$$

cos part will be zero as $\frac{1}{r} \approx 0$

Now we know $\hat{k} = \hat{z}$

$$I(t) = \frac{dq}{dt}\hat{k} = -q_0\omega\sin\omega t\,\hat{k} \quad \text{as } q = q_0\cos\omega t$$



$$A(r,t) = \frac{\mu_0}{4\pi}\int_{-\frac{S}{2}}^{\frac{S}{2}} -\frac{q_0\omega\sin\omega\left(t-\frac{R}{c}\right)\hat{k}}{R}\,dz$$

considering current flowing through a small section of dz

As like V we can compute the expression and similarly putting the limit the value at center is

$$\boxed{A(r,\theta,t) = -\frac{\mu_0 P_0\omega}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\hat{k}}$$

See the ~~vertical~~ variation V is scalar potential so no unit vector for Direction
A is vector " so $\hat{k}$. Both contain retarded time terms. Both contain
Both are oscillating so both contain retarded time
Dipole moment.

In spherical coordinate $\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

Now we need $E = -\nabla V - \frac{\partial A}{\partial t}$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} = \frac{P_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r}$$

Griffith's book Pg-404 approx r^2 term by neglecting.

similarly $$\frac{\partial A}{\partial t} = -\frac{\mu_0 P_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$\text{So } E = -\frac{P_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r} + \frac{\mu_0 P_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= -\frac{P_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r} + \frac{\mu_0 P_0\omega^2}{4\pi}\cos\omega\left(t-\frac{r}{c}\right)\frac{\cos\theta}{r}\hat{r} - \frac{\mu_0 P_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\sin\theta\,\hat{\theta}$$

$$= \frac{P_0\omega^2}{4\pi}\cos\omega\left(t-\frac{r}{c}\right)\left(\frac{\cos\theta}{r}\right)\left[-\frac{1}{\epsilon_0 c^2} + \mu_0\right] - \frac{\mu_0 P_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\sin\theta\,\hat{\theta}$$

$$= (\quad)\left[-\frac{1}{\epsilon_0\frac{1}{\epsilon_0\mu_0}} + \mu_0\right] - (\quad)$$

$c^2 = \frac{1}{\mu_0\epsilon_0}$

$$E = -\frac{\mu_0 P_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta}$$

Variation of E along $\theta \Rightarrow$ Elevation plane

Now

$$\nabla \times A = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

Here A is f^n of r, θ, t no variation in ϕ so $A_\phi = 0$

$$\nabla \times A = \hat{\phi}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left[\frac{\omega}{c}\sin\theta\cos\omega\left(t-\frac{r}{c}\right) + \frac{\sin\theta}{r}\sin\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}$$

So $B = \nabla \times A = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\phi}$ $\boxed{\text{It is in Azimuth Plane}}$

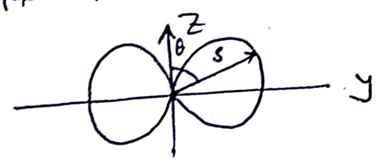
Now Both E & B contains single ω which represents monochromatic radiation. Both E & B have same phase. They are travelling in radial Direction. Both E & B mutually $\perp^r$ to each other which is denoted by $\hat{\theta}$ & $\hat{\phi}$. So it is the electromagnetic wave in free space.

And $E_0 / B_0 = c$

The Amplitude of E & B decreases by $1/r$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\right]^2 \hat{r}$$

Note if $\theta = 0$ $\sin 0^\circ = 0^\circ$ and hence $S = 0$. So it means that no radiation along the axis of dipole. For $\theta = 90^\circ$ max radiation.



Antenna Characteristic

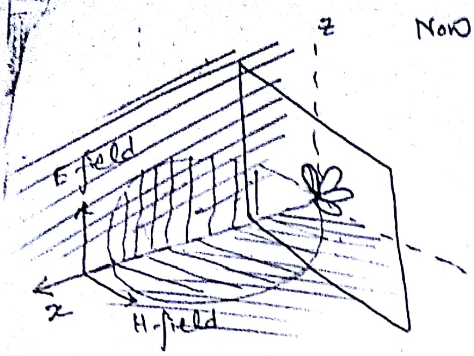
Radiation Pattern or antenna pattern is a mathematical function or graphical representation of the radiation properties of the antenna as a function of coordinates.

Isotropic Radiator → It is a hypothetical lossless antenna having equal radiation in all direction. Although it is ideal but not physically realizable.

Directive antenna → In practical this kind of antenna has the property of radiating or receiving em wave effectively from a certain direction than other direction.

Omnidirectional → Having pattern such that non directional pattern in a given plane and a directional pattern in any orthogonal plane. So we can have a similarity of omni pattern with isotropic pattern but on a certain plane.

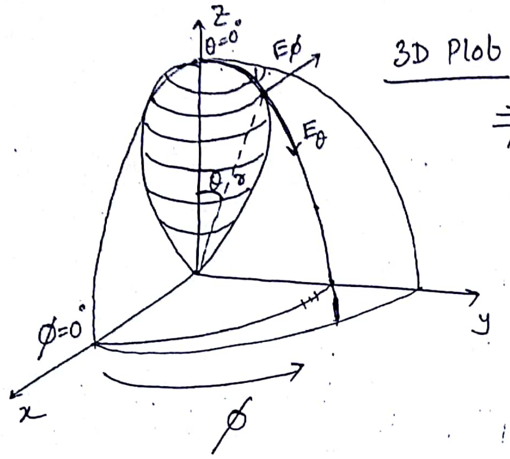
Principal Pattern :- Any antenna can be characterized by it's radiation pattern. Radiation pattern basically comprises of Field patterns. So by observing the nature of field patterns we can characterized the antenna pattern/nature. Thus the performance is often described in terms of principal E and H plane. An E plane is defined the plane containing the electric field vector and the direction of maximum radiation. Similarly H plane as plane containing the magnetic field vector and direction of maximum radiation.



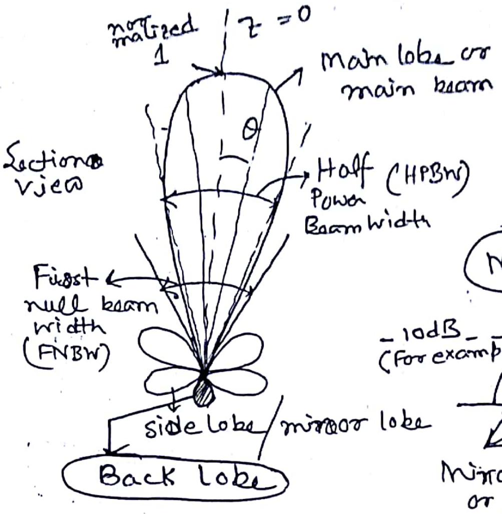
Now

E field max contains in XZ plane
Direction of max rad. in XZ "
So E plane is XZ plane
Similarly H plane is XY plane.

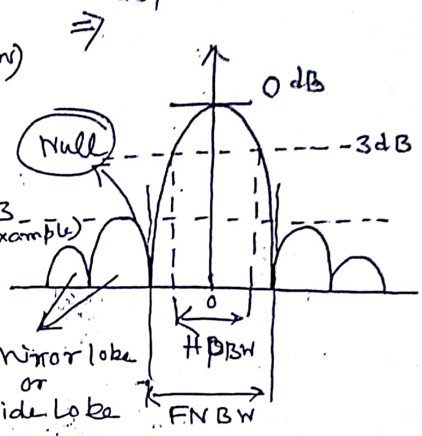
Pattern :-



3D Plot ⇒ Sectional view



⇒ dB pattern



The above figure shows the field intensity pattern in different form. Here r is propotional to field intensity at a certain distance from antenna in the direction θ and φ. The pattern has a main lobe maximum at z direction for θ = 0° and some other minor lobe in other direction. There are some nulls in which no radiation. So in general we need three plot to describe any field intensity at any 3D location.

① The θ component (Elevation) $E_\theta(\theta, \phi)$, ② φ component (Azimuth) $E_\phi(\theta, \phi)$, ③ ∠ between E_θ & E_ϕ

Now dividing a field component by its maximum value we get normalized field pattern.

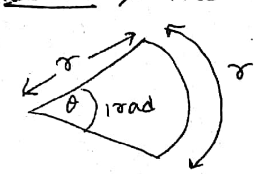
thus
$$E_\theta(\theta, \phi)_n = \frac{E_\theta(\theta, \phi)}{E_\theta(\theta, \phi)_{max}}$$

Pattern can also be expressed in terms of power/area or Poynting vector $S(\theta, \phi)$. Normalizing this Power/area w.r.t its maximum value gives $\boxed{\text{normalized power pattern,}}$ as a function of θ, φ and the max value is 1.

∴ Normalized Power pattern ⇒
$$P_n(\theta, \phi) = \frac{S(\theta, \phi)}{S(\theta, \phi)_{max}}$$

Any normalized power described in dB as $dB = 10 \log_{10} P_n(\theta, \phi)$ can give dB scale power pattern. This will enhance minor lobe detail. In the figure XZ and YZ are called principal plane pattern. Back lobe is a special case of minor lobe where some of the power radiated back towards antenna/ towards source. 3dB Beam width in dB scale (half power, 0.707) gives HPBW. It is a quantity described the beam shape – whether the beam is broad or narrow. FNBW → First null beam width describes the beam width between the null (First). First null is the first zero value next to principal maxima. It is not mandatory to have multiple null.

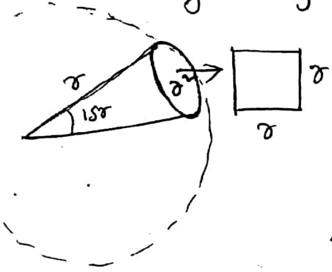
Radian → This is a measure of a plane angle.

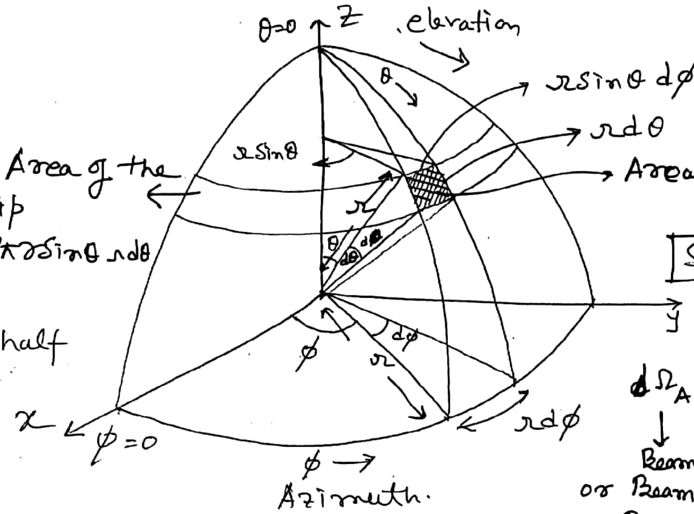


One radian is defined as the plane angle with its vertex at the center of a circle of radius r that is subtended by an arc whose length is r. Since circumference of a circle is $C = 2\pi r$. Therefore 2π rad is the ~~length of arc for full circle and this~~ 2π radian angle for full circle.

Steradian

The measure of a solid angle is steradian. One steradian is defined as the solid angle with its vertex at the center of sphere of radius r that is subtended by a spherical surface area equal to that of a square with each side of length r. Thus for a complete sphere 4π is the steradian angle.





Area of the strip $= 2\pi r \sin\theta \, r d\theta$

$d\Omega$ = Solid angle subtended by area dA

Area $dA = r \sin\theta \, d\phi \cdot r d\theta$
$= r^2 \sin\theta \, d\theta \, d\phi$

$\boxed{\text{Solid angle}} = \frac{dA}{r^2}$

$d\Omega = \sin\theta \, d\theta \, d\phi$

$$d\Omega_A = \int_0^{2\pi}\int_0^{\pi} \boxed{P_n(\theta,\phi)} \, d\Omega$$

Beam area or Beam solid angle ⇒ normalized Power Pattern over a sphere

Solid angle can be described in terms of angle subtended by the half power points of the main lobe in two principal plane

$$\boxed{\Omega_A = \theta_{HP}\,\phi_{HP}}$$

Radiation Intensity :- Radiation intensity is given by the power radiated from an antenna per unit solid angle. Unit of Radiation intensity is watt / steradian.

$$P_n(\theta,\phi) = \frac{U(\theta,\phi)}{U(\theta,\phi)_{max}}$$ where $U(\theta,\phi)$ is radiation intensity.

Total power is obtained by integrating the radiation intensity over the entire solid angle 4π

$$P_{rad} = \oint_{\Omega} U \, d\Omega = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$$

Radiation Intensity for an Isotropic source.

For an isotropic source (full spherical power pattern) intensity is independent of θ and ϕ.

so $$P_{rad} = \iint_0^{2\pi\ \pi} U \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} U d\Omega = U \int_0^{2\pi}\int_0^{\pi} d\Omega = 4\pi U$$

∴ $\boxed{U = \frac{P_{rad}}{4\pi}}$ → for isotropic source. $\boxed{\text{we denote it as } U_0}$

Directivity :- Directivity of an antenna is defined as the ratio of radiation intensity in a given direction from an antenna to the radiation intensity averaged over all direction. The denominator is nothing but radiation intensity of isotropic radiator.

$$D = \frac{U}{U_0} = \frac{4\pi U}{P_{rad}}$$

If the direction is not specified it implies that max radiation intensity or max Directivity is $D_0 = D_{max} = \frac{U|_{max}}{U_0} = \frac{4\pi U_{max}}{P_{rad}}$ ⇐ For non isotropic source

For isotropic source U_{max}, U, U_0 are all equal.

In a more general form we know $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi \, U_{max}}{\int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi} = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta \, d\theta \, d\phi \,/\, U(\theta,\phi)_{max}}$$

$$= \frac{4\pi}{\left[\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta \, d\theta \, d\phi\right] / U(\theta,\phi)_{max}} = \frac{4\pi}{\Omega_A}$$

where $\Omega_A = \frac{1}{U(\theta,\phi)_{max}} \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \underbrace{U_n(\theta,\phi)}_{\boxed{\text{this term constant under this assumption}}} \sin\theta \, d\theta \, d\phi$

$U_n(\theta,\phi)$ = normalized radiation intensity $\frac{U(\theta,\phi)}{U(\theta,\phi)_{max}}$

Ω_A is the beam solid angle, through which all the power of the antenna will flow if its radiation intensity is constant, for all angle within Ω_A. Beam solid angle is equal to the product of half power beamwidth in two $\perp^r$ plane.

$$\therefore D_0 = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\theta_{HP}\,\phi_{HP}} \quad \text{in radian.}$$

Beam Efficiency

Total Beam solid angle also known as beam area Ω_A consists of main beam area Ω_M + the minor lobe beam area Ω_m.

$$\therefore \Omega_A = \Omega_M + \Omega_m$$

Beam efficiency $\varepsilon_M = \frac{\Omega_M}{\Omega_A}$ and $\varepsilon_m = \frac{\Omega_m}{\Omega_A}$

And $\varepsilon_M + \varepsilon_m = 1$ (normalized)

$d\Omega$ = Solid angle subtended by area dA = $r^2 \sin\theta \, d\theta \, d\phi = r^2 d\Omega$

Ω_A = Beam solid angle or beam area = normalized power pattern over a sphere

$$\Omega_A = \int_0^{2\pi}\int_0^{\pi} P_n(\theta,\phi) \, d\Omega \qquad d\Omega = \sin\theta \, d\theta \, d\phi$$

$$\text{Directivity} = \frac{\text{Max Rad intensity}}{\text{Rad intensity Avg}} = \frac{\text{Max Power radiated in direction} \,/\, \text{solid angle through which Power rad}}{\text{Avg Power Rad} \,/\, \text{Angle subtended by a sphere}}$$

$$= \frac{P_{max}}{\text{Solid angle}} \times \frac{4\pi}{\text{Avg Power}} = \frac{4\pi}{\frac{\text{Avg P}}{P_{max}} \times \text{Solid angle through which max rad}}$$

normalized Power

$$= \frac{4\pi}{\Omega_A}$$

* Think what will happen for D in relativistic and non relativistic Domain.

Antenna Gain

Antenna gain is closely related with Directivity. But it also takes into account the antenna efficiency along with the directional property. Absolute gain or power gain in a given direction is defined as the ratio of the radiation intensity in a given direction to the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically. So unlike the defn of Directivity which tells about Power radiated isotropically irrespective to accepted power, Gain relates the Power radiated with power accepted.

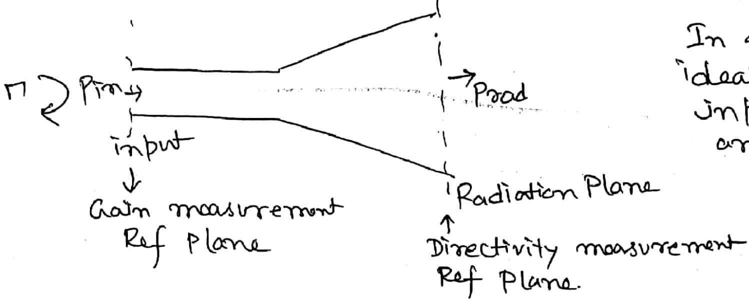
$$\therefore \quad \text{Gain} = 4\pi \times \frac{\text{Radiation intensity in a given direction}}{\text{total input Power}}$$

← compare for Directivity it is Prad

$$= 4\pi \times \frac{U(\theta,\phi)}{P_{in}} \quad \text{(Dimension less)}$$

In most cases relative gain is defined as Ratio of the Power gain in a given direction to the power gain of a reference antenna in its reference direction. Power input must be same for both antennas. In general the ref antenna is dipole or horn whose gain is known. It treats as lossless isotropic source.

When the direction is not stated, the power gain is usually taken as maximum radiation direction.



In general $P_{rad} = P_{in}$ for an ideal antenna but due to input mismatch and conductor and ~~direct~~ dielectric loss

$$\boxed{P_{rad} = \varepsilon_{ant} \times P_{in}}$$

where ε_{ant} is antenna radiation efficiency.

In general

$$G(\theta,\phi) = \varepsilon_{ant}\, D(\theta,\phi)$$

or

$$G_o = G(\theta,\phi)_{max} = \varepsilon_{ant}\, D(\theta,\phi)_{max} = \varepsilon_{ant} D_o$$

ε_{ant} comprises of primarily Γ (reflection losses due to mismatch) and I^2R losses.

Beamwidth, HPBW & Resolution of antenna :-

Often the term beamwidth is used to describe the angle between the two points on the pattern such as angle between the 10 dB points.

But in general beamwidth are commonly related with 3dB Beamwidth or Half Power Beam width (HPBW).

It is often used to provide a trade off between it and SLL. If Beamwidth decreases (narrow beam) the side lobe increases and vice versa. Beamwidth also describes the resolution capabilities of the antenna to distinguish between two adjacent radiating source. In general resolution capability of an antenna to distinguish between two source is equal to FNBW/2. Again ~~to~~ an approximation has generally made for design is $HPBW = \frac{FNBW}{2}$.

~~Input impedance → Balanis Pg 73 – 77~~

Antenna Radiation Efficiency :- As discussed earlier every antenna has some amount of conductor loss, dielectric loss. Antenna Radiation efficiency relates with this kind of losses with Radiation Resistance.

$$e_{ff} = \frac{R_r}{R_L + R_r}$$

Ratio of Power delivered to radiation resistance R_r to the power delivered to R_r & R_L.

Antenna in Receiving Mode

An antenna in receiving mode is used to collect EM wave and extract power from them. Thus for each antenna an equivalent length or an equivalent area is required to define. This equivalent quantity are used to describe the receiving nature of antenna.

Effective Length :- (voltage Relate) Effective length of an antenna (~~linear or apeture~~) is a quantity that is used to determine the voltage induced on the open ckt terminal of the antenna when a wave impinges upon it. The vector effective length l_e for an antenna usually a complex vector involving spherical coordinate system

$$l_e(\theta,\phi) = \hat{a}_\theta\, l_\theta(\theta,\phi) + \hat{a}_\phi\, l_\phi(\theta,\phi)$$ → It can also termed as effective height.

This quantity is related with field inside the wave.

$$l_e = f^{n}(E) \quad \& \quad E = \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi$$

If V_{oc} is the open circuit voltage in the terminal of antenna in receiving mode then

$$V_{oc} = E \cdot l_e$$

Please note about the polarization (The above V_{oc} for linearly Polarized)

Antenna Equivalent Area (Power Related) :- An antenna can be described interms of equivalent area. This is used to describe the power capturing characteristic of antenna when a wave impinges on it. One of the equivalent area is effective area which is defined as ratio of available power at the terminal of Rx antenna to the power flux density of a plane wave incident on antenna from that direction.

$$A_e = \frac{P_T}{W_i}$$

A_e = effective area (m^2)
P_T = Power delivered to load (W)
W_i = Power density of incident wave (W/m^2)

In general all the power that is intercepted or collected by an antenna is not delivered to the load. Under conjugate matching only half of the power is delivered to load. Other half is scattered and dissipated as heat. So in addition to effective area there are scattering area, loss area and capture area.

Scattering area Related with Radiation resistance
Loss " " " loss in terms of heat.
Capture area " all terms of R_L, R_r

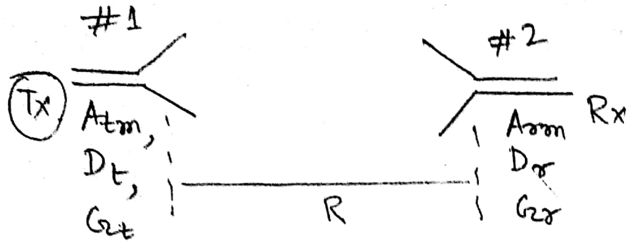
Capture area = Effective area + Scattering area + Loss area.

Now Apeture efficiency ϵ_{ap} of an antenna is defined as maximum effective area A_{em} to its physical area

$$\epsilon_{ap} = \frac{A_{em}}{A_p} = \frac{\text{Max } A_e}{\text{Physical Area}}$$

$\underbrace{A_{em} \neq A_p \text{ so } \epsilon_{ap} \text{ can not be } 100\%}_{\text{Aperture antenna}}$

Directivity or Gain vs maximum Effective area.



Antenna 1 → transmitter.
Antenna 2 → Receiver
Antenna 1 → choose as isotropic
so it radiated power density $W_0 = \dfrac{P_t}{4\pi R^2}$
where P_t total Radiated power.

Now actual power density if we change isotropic to a directive antenna.

$$W_t = W_0 D_t = \frac{P_t D_t}{4\pi R^2}$$

Power collected by Rx antenna is

$$P_r = W_t \times A_r = \frac{P_t D_t A_r}{4\pi R^2}$$

$$\therefore \; D_t A_r = \frac{P_r}{P_t}(4\pi R^2)$$

For 2 as tx & 1 as Rx (vice versa) $\Rightarrow D_r A_t = \dfrac{P_{rt}}{P_t}(4\pi R^2)$

$$\therefore \; \frac{D_t}{A_t} = \frac{D_r}{A_r} \Rightarrow \cancel{D/A_{eff}}$$

$$\Rightarrow \frac{D_{0t}}{A_{tm}} = \frac{D_{0r}}{A_{rm}}$$ | D_0, A_m are for max directivity, max effective area.

For our initial assumption of isotropic $D_{0t} = 1$ and $A_{tm} = \dfrac{A_{rm}}{D_{0r}}$.

Maximum effective area of an isotropic source is equal to the ratio of max effective area to max directivity of any other source

Remember → $\boxed{A_{em} = \dfrac{\lambda^2}{4\pi} D_0}$ → ≈ $\boxed{A_{em} = e_{cd} \dfrac{\lambda^2}{4\pi} D_0}$

lossless — Lossy

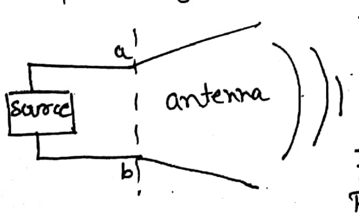
Again $G = KD$ ∴ $A_{em} = e_{cd} \dfrac{\lambda^2}{4\pi} \dfrac{G}{K}$

$K = \dfrac{A_p}{A_{em}}$

$$\Rightarrow \; G = \frac{4\pi}{\lambda^2} A_p$$

Input Impedance :-

Input impedance is defined as the impedance presented by an antenna at its terminal or the ratio of voltage to current at a pair of terminal or the ratio of appropriate components of electric or magnetic field. Input impedance ≡ at input terminal of antenna

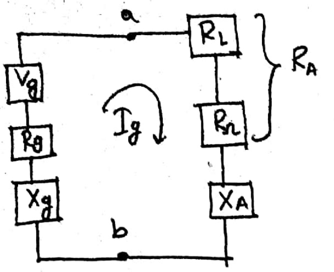


Here the input terminal is inbetween a–b. The ratio of voltage to current at (a–b) with no load attached (open-end antenna) can be defined as $Z_A = R_A + jX_A$

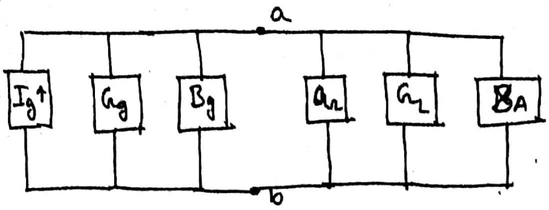
Z_A = antenna imp at a–b
R_A = antenna resistance at a–b
X_A = antenna reactance at a–b.

In general we have observed that $R_A = R_r + R_L$
R_r = radiation resistance
R_L = loss resistance.

If now we think of maximum power transfer case then this antenna impedance should be complex conjugate of generator impedance which is $Z_g = R_g + jX_g$
Now, to find the amount of power delivered to R_r for radiation and amount dissipated in R_L as heat $(I^2 R_L/2)$ we need to find out the total current in the equivalent circuit of antenna.



thevenin eqv



$$\therefore I_g = \frac{V_g}{Z_{total}} = \frac{V_g}{Z_A + Z_g} = \frac{V_g}{(R_r+R_L+R_g)+j(X_A+X_g)}$$

$$|I_g| = \frac{|V_g|}{\sqrt{(R_L+R_r+R_g)^2+(X_A+X_g)^2}} \Rightarrow V_g \text{ is Peak generator voltage.}$$

Now Power delivered to the antenna for perfect radiation is

$$P_r = \frac{1}{2}|I_g|^2 R_r = \frac{|V_g|^2}{2}\frac{R_r}{(R_r+R_L+R_g)^2+(X_A+X_g)^2} \text{ Watt.}$$

and loss or heat dissipated

$$P_L = \frac{1}{2}|I_g|^2 R_L = \frac{|V_g|^2}{2}\frac{R_L}{(R_r+R_L+R_g)^2+(X_A+X_g)^2} \text{ Watt.}$$

Now the remaining power will be dissipated as heat on the internal resistance R_g of generator.

$$P_g = \frac{|V_g|^2}{2}\frac{R_g}{(R_r+R_L+R_g)^2+(X_A+X_g)^2} \text{ Watt.}$$

From maximum power transfer theorem

$$R_r + R_L = R_g \quad \text{and} \quad X_A = -X_g$$

$\therefore$ Putting the condition of Conjugate Complex

$$P_r = \frac{|V_g|^2}{2}\frac{R_r}{4(R_L+R_r)^2} = \frac{|V_g|^2}{8}\frac{R_r}{(R_r+R_L)^2}$$

$$P_L = \frac{|V_g|^2}{8}\frac{R_L}{(R_r+R_L)^2} \quad \& \quad P_g = \frac{|V_g|^2}{8}\frac{R_g}{(R_r+R_L)^2}$$

Again as $R_r+R_L=R_g$ then

$$P_g = \frac{|V_g|^2}{8}\cdot\frac{R_g}{R_g^2} = \frac{V_g^2}{8R_g}$$

Now

$$\boxed{P_r + P_L = \frac{|V_g|^2}{8}\left(\frac{R_r+R_L}{(R_r+R_L)^2}\right) = \frac{|V_g|^2}{8R_g} = P_g}$$

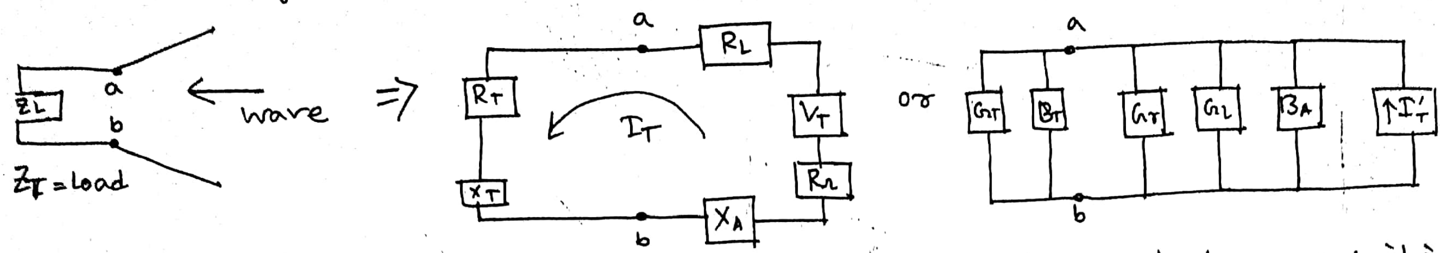
Power supplied by generator of voltage V_g and current I_g ~~is~~ during conjugate matching is

$$P_s = \frac{1}{2}V_g I_g^* = \frac{1}{2}V_g\left[\frac{V_g^*}{2(R_r+R_L)}\right] \qquad \begin{aligned} X_A &= -X_g \\ R_g &= R_r+R_L \end{aligned}$$

$$\boxed{P_s = \frac{|V_g|^2}{4}\frac{1}{R_r+R_L} \text{ Watt}}$$

It means the power provided by generator, half is dissipated as heat in the internal resistance (R_g) and other half is delivered to antenna during conjugate matching. Of the power that is delivered to antenna, part is radiated through the radiation resistance and other is dissipated as heat. Thus all those kind of losses at antenna affects the efficiency of antenna. If antenna is loss less ($e_{cd}=1$) then half of total power supplied to antenna for radiation.

Equivalent circuit of antenna in receiving mode.-



During receiving mode, the incident wave impinges on antenna aperture and it induces a voltage V_T which is analogous to V_g of transmitting mode. Now Thevenin equivalent circuit has been changed. The procedure to calculate ~~indivi~~ individual component are same as like transmitting mode. In receiving mode maximum power transfer happens at

$$R_L + R_l = R_T \text{ and } X_A = -X_T.$$

Thus
$$P_T = \frac{|V_T|^2}{8}\left[\frac{R_T}{(R_r+R_L)^2}\right] = \frac{|V_T|^2}{8R_T}$$

$$P_r = \frac{|V_T|^2}{2}\left[\frac{R_r}{4(R_r+R_L)^2}\right] = \frac{|V_T|^2}{8}\left[\frac{R_r}{(R_r+R_L)^2}\right]$$

$$P_L = \frac{|V_T|^2}{8}\left[\frac{R_L}{(R_r+R_L)^2}\right]$$

Similarly (as like P_s) $\quad P_c = \frac{1}{2}V_T I_T^* = \frac{1}{2}V_T\left[\frac{V_T^*}{2(R_r+R_L)}\right] = \frac{|V_T|^2}{4}\left(\frac{1}{R_r+R_L}\right)$

The Power delivered to P_r is called re-radiated Power.
Thus under conjugate matching total power collected or captured (P_c), half is delivered to load R_T and the other half is scattered or reradiated through R_r and dissipated as heat through R_L.

Field Configuration of Hertzian Dipole.

To know antenna characteristics, the knowledge of current distribution is required.
We know from the magnetic vector Potential approach $\quad A = \frac{\mu}{4\pi}\int_v \frac{J}{R}e^{-jKR}\,dv'$

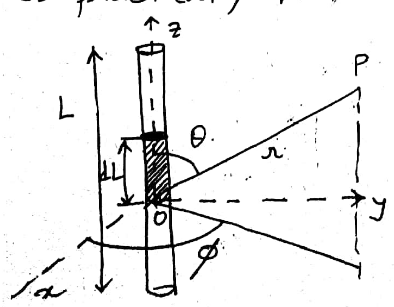
But as we have discussed that this only justifies the space coordinates.
Actual magnetic vector potential should be described both in time and space by

$$A(r,t) = \frac{\mu}{4\pi}\int_v \frac{J(r,t)}{R}\,dv$$

From the knowledge of retarded potential where the wave travels with finite velocity v can be used to describe a generalized form of A for a current source J as

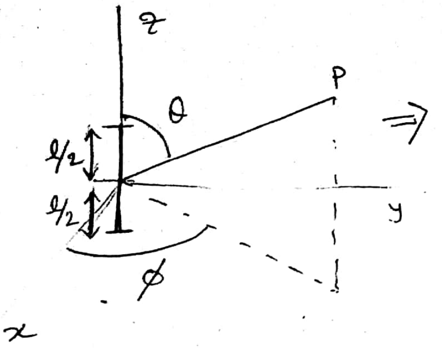
$$A(r,t) = \frac{\mu}{4\pi}\int_v \frac{J\left(r,\, t-\frac{R}{v}\right)}{R}\,dv$$

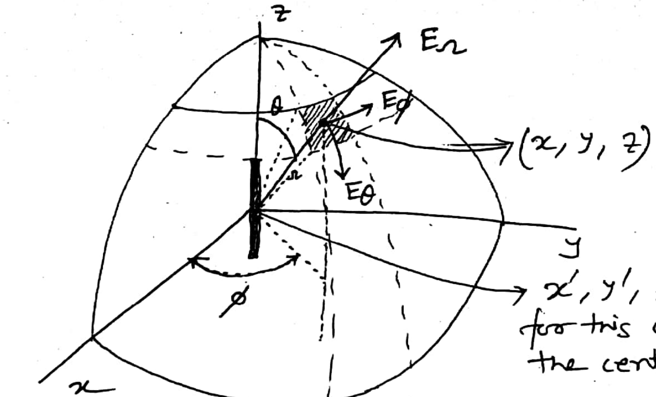
With this concept we are now deducing the field component of Hertzian Dipole
A Hertzian Dipole refers to an infinitesimal current carrying element which is practically not possible but useful for analysis of practical Dipole antenna



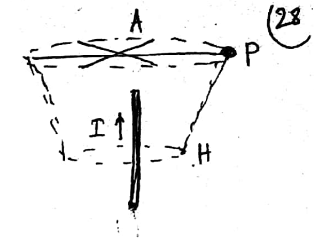
As shown by Hertzian Dipole, located by z axis in coordinate system. The length of Dipole is dL and diameter is negligible compared to l. For infinitesimal dipole we consider $l \ll \lambda$.
Current is assumed constant $\quad I(z) = \hat{a}_z I_0$

* Please note: this is a single discreate wire. It is not as twin wire dipole. That is why we assume current is uniform.





$\rightarrow x', y', z'$ if not at center for this case x', y', z' is at the center.



Assume that the current over the dipole is sinusoidal then with the concept of Retarded Potential at any point P located at r from origin

If x', y', z' is not at origin then
$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$A = \frac{\mu}{4\pi} \iiint \frac{\vec{J}\left(t - \frac{r}{v}\right)}{r} \, dv$$

| Note if J is ~~at~~ at origin $J(r, t) = J(t)$ ~~and~~ as J is at the center hence $J(r)$ dependency is cancelled.

where $t' = t - \frac{r}{v}$ | Due to time dialation.

r = Distance of point P from origin | otherwise R will be termed.

v = velocity of wave

we know that in a dipole antenna charge oscillates with applied sinusoidal signal. If we consider this oscillation of charge as $q_0 \sin \omega t$ then $\frac{dq}{dt} = I$. So current distribution on dipole is $I = I_0 \cos \omega t$ which can be approximated by $I = I_0 e^{j\omega t}$. Though this current is the primary source of radiation but using retarded potential we can write this $I_{retarded}$ as $I_0 e^{j\omega(t - r/v)}$.

So ~~that~~ total volume integral of current density can be given by

$$\int_v J(t') \, dv = I \, dl \cos \omega t'$$

where dl is the segment of length on which the total current is distributed and $t' = t - r/v$.

Now substituting the total current I on A_z we get
$$A_z = \frac{\mu I \, dl \cos \omega t'}{4\pi r}$$

Again from A we can obtain H by $\nabla \times A = \mu H$.

There is a relationship by which the A in (x, y, z) can be converted to (r, θ, ϕ) so

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

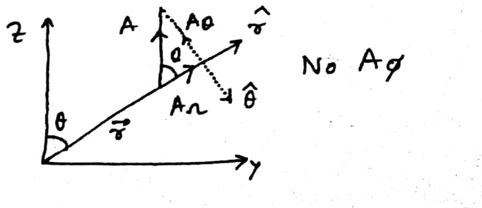
As for this case A has only z component so $A_x = A_y = 0$ from the above matrix we obtain $\boxed{A_r = A_z \cos\theta}$, $\boxed{A_\theta = -A_z \sin\theta}$, $A_\phi = 0$

Now
$$\nabla \times A = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{a}_r - \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{a}_\phi$$

Assume there is symmetry in $\phi \Rightarrow$ No variation in ϕ.

$$\nabla \times A = \hat{a}_\phi \frac{1}{r}\left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

How we visualize A components.



Radiated Power from Hertzian Dipole.

Power radiated from Dipole are calculated by using Poynting theorem. Power density or Power radiated per unit area is given by Poynting vector (P)

Instantaneous Power (W/m²) $\boxed{P = E \times H}$ · for time avg this is $\frac{1}{2} E \times H^*$

Now $P_r \hat{a}_r + P_\theta \hat{a}_\theta + P_\phi \hat{a}_\phi = \left[\{ E_r \hat{a}_r + E_\theta \hat{a}_\theta + E_\phi \hat{a}_\phi \} \times \{ H_r \hat{a}_r + H_\theta \hat{a}_\theta + H_\phi \hat{a}_\phi \} \right]$

Now from the previous equations we get

$$P_r \hat{a}_r + P_\theta \hat{a}_\theta + P_\phi \hat{a}_\phi = \left[\{ E_r \hat{a}_r + E_\theta \hat{a}_\theta \} \times H_\phi \hat{a}_\phi \right]$$

$E_r H_\phi \; \hat{a}_r \times \hat{a}_\phi \} - \hat{a}_\theta$

$E_\theta H_\phi \; \underbrace{\hat{a}_\theta \times \hat{a}_\phi}_{\hat{a}_r}$

$$\Rightarrow \quad P_r = E_\theta H_\phi$$
$$P_\theta = -E_r H_\phi$$
$$P_\phi = 0$$

Now by substituting E_r, E_θ and H_ϕ we can get each component.
First by substituting E_r and H_ϕ we can get

$$P_\theta = -\left[\left\{ \frac{2 I dL \cos\theta}{4\pi\varepsilon_0} \left[\frac{\cos\omega t'}{r^2 v} + \frac{\sin\omega t'}{\omega r^3} \right] \right\} \times \left\{ \frac{I dL \sin\theta}{4\pi} \left[-\frac{\omega \sin\omega t'}{r v} + \frac{\cos\omega t'}{r^2} \right] \right\} \right]$$

$$= \left[\frac{I^2 dL^2 (2\sin\theta\cos\theta)}{16\pi^2\varepsilon_0} \right] \left[\frac{\omega (2\sin\omega t' \cos\omega t')}{2 v^2 r^3} - \frac{\cos^2 \omega t'}{v r^4} + \frac{\sin^2 \omega t'}{v r^4} - \frac{\sin 2\omega t'}{2\omega r^5} \right]$$

$$= \left[\frac{I^2 dL^2 \sin 2\theta}{16\pi^2\varepsilon_0} \right] \left[\frac{\omega \sin 2\omega t'}{2 v^2 r^3} - \frac{\cos 2\omega t'}{v r^4} - \frac{\sin 2\omega t'}{2\omega r^5} \right]$$

Now average Power P_θ over a complete cycle is zero as the avg value of $\sin 2\omega t'$, $\cos 2\omega t'$ over complete cycle is zero. So net radiation in θ direction ~~and~~ is zero. Power flows back and forth in this direction.

Now For P_r

$$P_r = \left[\left\{ \frac{I dL \sin\theta}{4\pi\varepsilon_0} \left[-\frac{\omega \sin\omega t'}{v r} + \frac{\cos\omega t'}{v r^2} + \frac{\sin\omega t'}{\omega r^3} \right] \right\} \times \left\{ \frac{I dL \sin\theta}{4\pi} \left[-\frac{\omega \sin\omega t'}{r v} + \frac{\cos\omega t'}{r^2} \right] \right\} \right]$$

$$= \left[\frac{I^2 dL^2 \sin^2\theta}{16\pi^2\varepsilon_0} \right] \left[\frac{\omega^2}{2 r^2 v^3} - \frac{\omega^2 \cos 2\omega t'}{2 r^2 v^3} - \frac{\omega \sin 2\omega t'}{r^3 v^2} + \frac{\cos 2\omega t'}{r^4 v} + \frac{\sin 2\omega t'}{2\omega r^5} \right]$$

$$\therefore \text{ Avg value of } P_r = \left[\frac{I^2 dL^2 \sin^2\theta}{16\pi^2\varepsilon_0} \right] \left[\frac{\omega^2}{2 r^2 v^3} \right] = \frac{\omega^2 I^2 dL^2 \sin^2\theta}{32\pi^2 r^2 v^3 \varepsilon_0}$$

If v = velocity of light and ε_0 = free space permittivity then

$$\langle P_r \rangle = \frac{\omega^2 I^2 dL^2 \sin^2\theta}{32\pi^2 r^2 c^3 \varepsilon_0} = \frac{1}{2c\varepsilon_0} \left(\frac{\omega I dL \sin\theta}{4\pi r c} \right)^2 = \frac{\eta_0}{2} \left(\frac{\omega I dL \sin\theta}{4\pi r c} \right)^2$$

where $\eta_0 = 377\,\Omega$ or 120π.

$$\therefore \quad \langle P_r \rangle = \boxed{60\pi \left(\frac{\omega I dL \sin\theta}{4\pi r c} \right)^2}$$

This is the average power flow radiated by Hertzian Dipole in radial direction

In the previous calculation all the three terms includes all the effects of r and hence the Power obtained can be the power in far field if r is large and can be near field power if r < very small.

We can write

$$E_\theta = \frac{I\,dL\sin\theta}{4\pi\epsilon_0}\left[-\frac{\omega\sin\omega t'}{v^2 r}\right] \quad \text{and} \quad H_\phi = \frac{I\,dL\sin\theta}{4\pi}\left[-\frac{\omega\sin\omega t'}{r v^2}\right] \text{ for } r \text{ large}$$

Total Radiated Power

As obtained from $\langle P_T \rangle$ Dipole can radiate power in the radial direction. This radial direction can be around the antenna and in any direction. Hence to find out total radiated Power, this average power can be integrated over the entire sphere surface.

$$\therefore\ P_T = \int_{surface} P_r\, ds. \quad \Big|\ ds = r^2 \sin\theta\, d\theta\, d\phi$$

$$P_T = \int_{surface}\left[\frac{\eta_0}{2}\left(\frac{\omega I\,dL\sin\theta}{4\pi r c}\right)^2\right]\left[r^2 \sin\theta\, d\theta\, d\phi\right]$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{\eta_0}{2}\left(\frac{\omega I\,dL\sin\theta}{4\pi r c}\right)^2\right]\left[r^2\sin\theta\right]d\theta\, d\phi$$

$$= \left[\frac{\eta_0\,\omega^2 I^2 dL^2}{32\pi^2 r^2 c^2}\right] r^2 \int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi = \frac{\eta_0\,\omega^2 I^2 dL^2}{32\pi^{\cancel{2}}\cancel{r^2} c^2}(2\cancel{\pi})\int_0^{\pi}\sin^3\theta\, d\theta$$

$$= \frac{\eta_0\,\omega^2 I^2 dL^2}{16\pi c^2}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{\eta_0\,\omega^2 I^2 dL^2}{12\pi c^2}$$

By converting Peak current into rms we can get. $I = \sqrt{2}\, I_{eff}$.

$$\therefore\ P_T = \frac{\eta_0\,\omega^2 I_{eff}^2\, dL^2}{6\pi c^2} \quad \Big|\ \text{Now } \eta_0 = 120\pi \text{ and } \left(\frac{\omega}{c}\right)^2 = \left(\frac{2\pi f}{c}\right)^2 = \left(\frac{2\pi \cancel{c}}{\cancel{c}\lambda}\right)^2 = \frac{4\pi^2}{\lambda^2}$$

$$\Rightarrow P_T = \frac{120\pi\, I_{eff}^2\, dL^2}{6\pi} \times \frac{4\pi^2}{\lambda^2} = 80\pi^2\left[\frac{dL}{\lambda}\right]^2 I_{eff}^2$$

Now from the relationship between P_T and Radiation resistance R_r by I_{eff}

$$P_T = R_r I_{eff}^2 \qquad \therefore\ R_r = 80\pi^2\left[\frac{dL}{\lambda}\right]^2$$

In above calculation P_θ component does not contribute to any part in integral. As P_θ does not have any real component it does not have any real radiated Power. It does have some effect in reactive power. For a complete cycle it provides zero value as if in reactive components (Resonator) for a full cycle energy transfer from one to other.

So we can write
$$P_T = \underbrace{P_{rad}}_{\text{Rad } P \Rightarrow P_r} + \underbrace{2j\omega(W_m - W_e)}_{P_\theta \text{ direction.}}$$

* If $\left(\frac{L}{\lambda}\right) = \frac{1}{10}$ then $R_r = 7.9\,\Omega$, $\left(\frac{L}{\lambda}\right) = \frac{1}{100}$ then $R_r = 0.08\,\Omega$. Radiation resistance of small dipole ($l \leq \lambda/50$) is small.

* The reactance of an Infinitesimal dipole is capacitive. We have seen two open ended flared line construct the dipole. Since input impedance of open circuited transmission line of length $l/2$ from open end is $Z_{in} = -jZ_c \cot(\beta l/2)$. Then it always be -ve for $l \ll \lambda$.

In near field $E_r = \frac{2I\,dL\cos\theta}{4\pi\varepsilon_0}\left[\frac{\sin\omega t'}{\omega r^3}\right]$, $E_\theta = \frac{I\,dL\sin\theta}{4\pi\varepsilon_0}\left[\frac{\sin\omega t'}{\omega r^3}\right]$, $H_\phi = \frac{I\,dL\sin\theta}{4\pi}\left[\frac{\cos\omega t'}{r^2}\right]$

$P_r = E_\theta H_\phi \qquad P_\theta = -E_r H_\phi$

$$P_r = \frac{I\,dL\sin\theta}{4\pi\varepsilon_0}\left[\frac{\sin\omega t'}{\omega r^3}\right] \times \left[\left\{\frac{I\,dL\sin\theta}{4\pi}\left[\frac{\cos\omega t'}{r^2}\right]\right\}\right]$$

$$= \frac{I^2 dL^2 \sin^2\theta}{16\pi^2\varepsilon_0}\left[\frac{\sin\omega t'\cos\omega t'}{\omega r^5}\right] = \frac{I^2 dL^2\sin^2\theta}{32\pi^2\varepsilon_0}\,\frac{\sin 2\omega t'}{\omega r^5}$$

$$P_\theta = -\left[\frac{2I\,dL\cos\theta}{4\pi\varepsilon_0}\left\{\frac{\sin\omega t'}{\omega r^3}\right\} \times \frac{I\,dL\sin\theta}{4\pi}\left\{\frac{\cos\omega t'}{r^2}\right\}\right]$$

$$= -\left[\frac{2\,I^2 dL^2\sin\theta\cos\theta}{16\pi^2\varepsilon_0}\left\{\frac{\sin\omega t'\cos\omega t'}{\omega r^5}\right\}\right] = -\frac{I^2 dL\sin\theta\cos\theta}{16\pi^2\varepsilon_0}\,\frac{\sin 2\omega t'}{\omega r^5}$$

$\langle P_r\rangle$ and $\langle P_\theta\rangle$ both are 0 as over full time cycle $\sin 2\omega t' \to 0$ which justifies the reactive component.
But they have imaginary power ~~com~~

Directivity

we have obtained $\langle P_r\rangle = \frac{\eta_0}{2}\left(\frac{\omega I\,dL\sin\theta}{4\pi r c}\right)^2$

Now From def^n of Radiation intensity from average power ~~$W = r^2 P_{rad}$~~

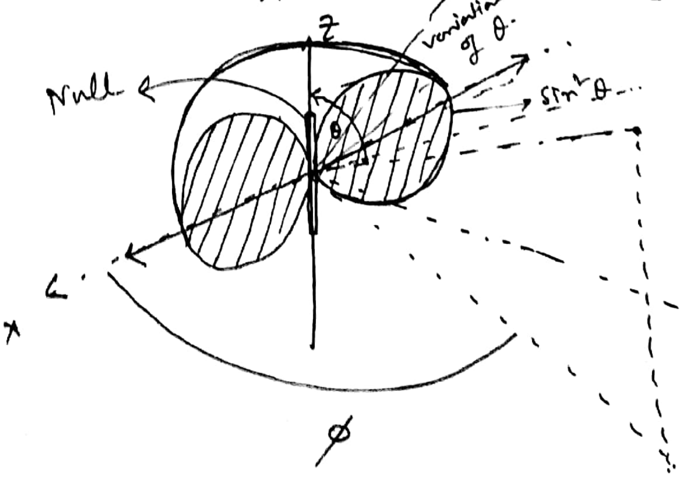
$U = r^2\langle P_r\rangle = \frac{r^2\eta_0}{2}\left(\frac{\omega I\,dL\sin\theta}{4\pi r c}\right)^2$ | $U_{max} = \frac{r^2\eta_0}{2}\left(\frac{\omega I\,dL}{4\pi r c}\right)^2$ for $\theta = 90^\circ$

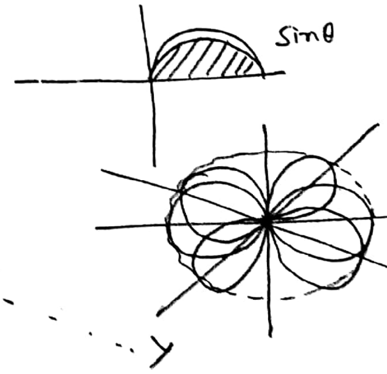
Now total Power $P_T = \frac{\eta_0\,\omega^2 I^2 dL^2}{12\pi c^2}$

$$\therefore \text{Directivity} = D_0 = 4\pi\frac{U_{max}}{P_T} = \frac{{}^2\cancel{4}\pi\,\frac{\cancel{r^2}\eta_0}{\cancel{2}}\left(\frac{\omega^2 I^2 dL^2}{16\pi^2\cancel{r^2}c^2}\right)}{\frac{\eta_0\,\omega^2 I^2 dL^2}{12\pi c^2}}$$

$$= \frac{\cancel{2}\cancel{\pi}\cancel{\eta_0}\,\cancel{\omega^2}\cancel{I^2}\cancel{dL^2}}{{}_8\cancel{16}\,\cancel{\pi^2}\,\cancel{c^2}} \times \frac{12\,\cancel{\pi}\cancel{c^2}}{\cancel{\eta_0}\cancel{\omega^2}\cancel{I^2}\cancel{dL^2}} = \frac{3}{2}$$

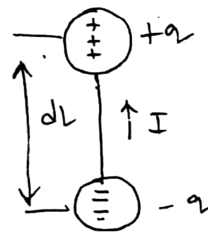
and A_{em} = maximum effective aperture $= \left(\frac{\lambda^2}{4\pi}\right) D_0 = \frac{3\lambda^2}{8\pi}$



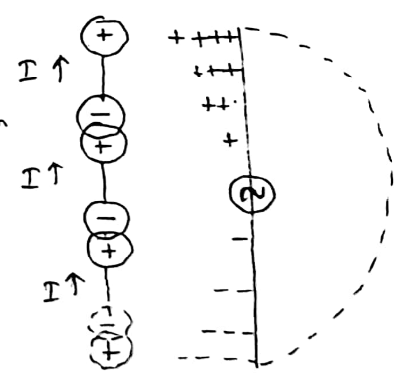


Please Remember :- If you check carefully you will ~~a~~ observe that Field component is nothing but Fourier Transform of Finite Length current distribution. Similarly inverse F Transform will give current element.

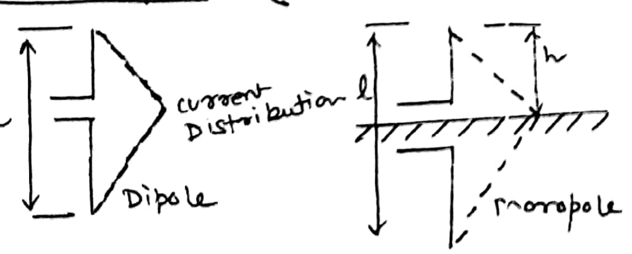
Remember | When we are computing a small current element I through dL we assume that this current is constant through out the wire. Here the I is oscillating. Now with the equation of continuity and to have current oscillation there must be some accumulation of charge at the ends of element. This is given by $\frac{dq}{dt} = I\cos\omega t$. It means charge at one end is decreasing and at the other end is increasing. The effect has been visible through osillating current flow. But there can not be an isolated current element in a single wire (Need a close ckt). Hence to obtain a physical approximation of isolated current element, one could terminate the current element in two small sphere on which charge could accumulate. If the wire is very thin, compared with the radius of the sphere, the current in the wire will be uniform. In addition, the radii of the sphere should be small compared with dL.



Now when the current element forms a part of complete circuit, there is no accumulation of charge at its ends if the current is uniform through out the circuit. As if there are small element and current is flowing from one element to other. Now as no accumulation of charges so we can expect no static charge which gives $1/r^3$. Thus there will be only induction and radiation field. We can imagine this as a line consisting of small dipoles. Each dipole cancels the next one by an equal amount of opposite charge. However if the current along the circuit is not uniform along length, then we can imagine chain of Dipole having slightly different amplitude. Here the adjacent dipole donot completely cancel and then there is accumulation of charge on the surface of wire.



Short Dipole (To make some kind of Practical antenna from hypothetical Hertzian Dipole)



A practical elementary dipole is the center fed antenna having a length that is very short in wavelengths. The current amplitudes on such antenna decreases uniformly over a maximum at center to zero at both ends. Thus field strength at every point are reduced to $\boxed{\text{one}^* \text{ half}}$ and the Power density will be reduced to one quarter.

Thus R_{rad} (short dipole) $= \frac{1}{4}\left[80\pi^2\left(\frac{L}{\lambda}\right)^2\right] = 20\pi^2\left(\frac{L}{\lambda}\right)^2 = 200\left(\frac{L}{\lambda}\right)^2 \Omega$

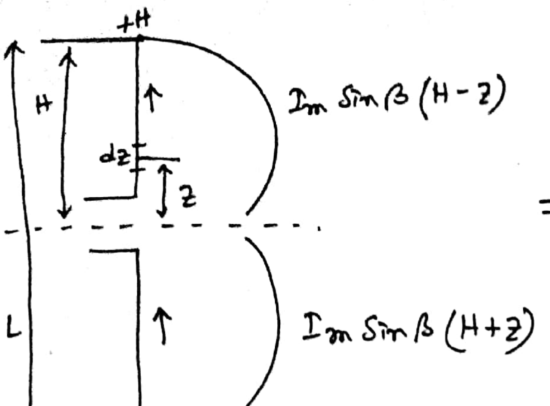
Similarly monopole can radiate half of total power radiated by short dipole. consider $l = 2h$.

$$\therefore R_{rad} \text{ (short monopole)} = \frac{1}{2}\, 200\left(\frac{L}{\lambda}\right)^2 = 100\left(\frac{L}{\lambda}\right)^2$$

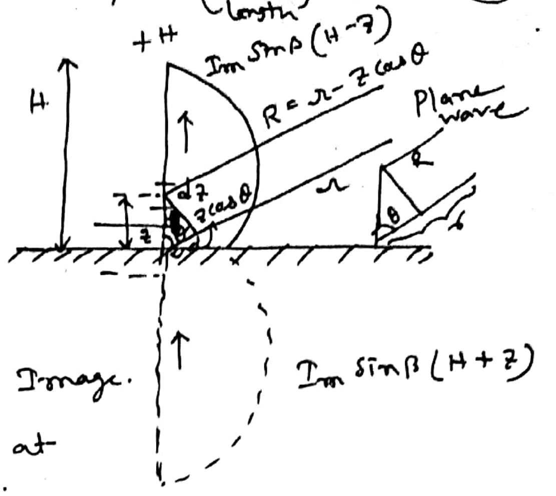
$$\therefore R_r \text{ (short monopole)} = 100\left(\frac{2h}{\lambda}\right)^2 = 400\left(\frac{h}{\lambda}\right)^2 \Omega$$

* Why half: For prev calculation I is uniform. But here it has a triangular distribution. Avg of I $= I_0 = 0.5\, I_m$.

Field Component and Power Radiation for Dipole and Monopole (Finite Length)



$\Rightarrow$ Monopole equivalent



$H = L/2$

Dipole radiates maximum in the plane which is $\perp^r$ to axis. Monopole is placed at a height H over perfect Reflecting plane.

Also considering current is not constant but varying sinusoidally. We can assume different dipole (small) with different intensity connected in series. The value of current is maximum at center and zero at two ends. Now the expression of current distribution in two half can be considered as

$I = I_m \sin\beta(H - z)$ for $z > 0$
$I = I_m \sin\beta(H + z)$ for $z < 0$

We need to find out vector potential at Point P which is located R distance away from $I\,dz$.

$$dA_z = \frac{\mu I}{4\pi R} e^{-j\beta R} dz \qquad \text{total } A_z = \frac{\mu}{4\pi}\int_{-H}^{H} \frac{I}{R} e^{-j\beta R} dz$$

$$\Rightarrow A_z = \frac{\mu}{4\pi}\int_{-H}^{0} \frac{I}{R} e^{-j\beta R} dz + \frac{\mu}{4\pi}\int_{0}^{H} \frac{I}{R} e^{-j\beta R} dz$$

$$= \frac{\mu}{4\pi}\int_{-H}^{0} \frac{(I_m \sin\beta(H+z))}{R} e^{-j\beta R} dz + \frac{\mu}{4\pi}\int_{0}^{H} \frac{(I_m \sin\beta(H-z))}{R} e^{-j\beta R} dz$$

From the Figure of monopole we have obtained that at point P with respect to origin the distance is r. But the current carrying element is z distance away. So with respect to origin R can be replaced by $r \Rightarrow r - z\cos\theta$. (In phase only)

Approximation:- For a radiation field we can imagine $R = r$. As if both the signal coming at same position for a distant observer. But for Phase factor $PL \to$ there is a phase difference between two signal reaching at P. so in num $R = r - z\cos\theta$

$$\therefore A_z = \frac{\mu}{4\pi}\int_{-H}^{0} \frac{I_m \sin\beta(H+z)}{r} e^{-j\beta(r - z\cos\theta)} dz + \frac{\mu}{4\pi}\int_{0}^{H} \frac{I_m \sin\beta(H-z)}{r} e^{-j\beta(r - z\cos\theta)} dz$$

$$A_z = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_{-H}^{0} \sin\beta(H+z)\, e^{+j\beta z\cos\theta} dz + \int_{H}^{0} \sin\beta(H-z)\, e^{j\beta z\cos\theta} dz\right]$$

Now For $\boxed{H = \frac{\lambda}{4}}$ $\quad \beta = \frac{2\pi}{\lambda} \quad \therefore \beta H = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2}$

$$\sin\beta(H+z) = \sin\left(\frac{\pi}{2} + \beta z\right) = \cos\beta z \quad \& \quad \sin\beta(H-z) = \sin\left(\frac{\pi}{2} - \beta z\right) = \cos\beta z$$

Putting in A_z

$$A_z = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_{-H}^{0} \cos\beta z\, e^{j\beta z\cos\theta} dz + \int_{0}^{H} \cos\beta z\, e^{j\beta z\cos\theta} dz\right]$$

Now $\displaystyle\int_{-H}^{0} \cos\beta z\, e^{j\beta z\cos\theta} dz = \int_{0}^{H} \cos(-\beta z)\, e^{-j\beta z\cos\theta} dz$ | Changing the limit

$$A_z = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_0^{H}\cos(-\beta z)e^{-j\beta z\cos\theta}\,dz + \int_0^{H}\cos(\beta z)e^{j\beta z\cos\theta}\,dz\right]$$

$$= \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_0^{H}\cos\beta z\, e^{-j\beta z\cos\theta}\,dz + \int_0^{H}\cos(\beta z)\, e^{j\beta z\cos\theta}\,dz\right]$$

$$= \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_0^{H}\cos\beta z\left[e^{j\beta z\cos\theta} + e^{-j\beta z\cos\theta}\right]dz\right]$$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$$= \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_0^{H}\cos\beta z\left[2\cos(\beta z\cos\theta)\right]dz\right] = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\int_0^{H}2\cos\beta z\cos(\beta z\cos\theta)\,dz$$

Now From $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ | Assume $A = \beta z$ & $B = \beta z\cos\theta$

$$A_z = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\int_0^{H}\left\{\cos\beta z(1+\cos\theta) + \cos\beta z(1-\cos\theta)\right\}dz\right]$$

Putting $H = \lambda/4$

$$A_z = \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\frac{\sin\beta z(1+\cos\theta)}{\beta(1+\cos\theta)} + \frac{\sin\beta z(1-\cos\theta)}{\beta(1-\cos\theta)}\right]_0^{\lambda/4}$$

$$= \frac{\mu I_m e^{-j\beta r}}{4\pi r}\left[\frac{(1-\cos\theta)\left\{\sin\frac{\pi}{2}(1+\cos\theta)\right\} + (1+\cos\theta)\left\{\sin\frac{\pi}{2}(1-\cos\theta)\right\}}{\beta(1-\cos^2\theta)}\right]$$ Putting limit of z

$$= \frac{\mu I_m e^{-j\beta r}}{\beta 4\pi r}\left[\frac{(1-\cos\theta)\left(\cos\left(\frac{\pi}{2}\cos\theta\right)\right) + (1+\cos\theta)\left(\cos\left(\frac{\pi}{2}\cos\theta\right)\right)}{\sin^2\theta}\right]$$

$$= \frac{\mu I_m e^{-j\beta r}}{4\pi r\beta}\left[\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right] = \frac{\mu I_m e^{-j\beta r}}{2\pi r\beta}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right)$$

Now $H = \frac{1}{\mu}(\nabla\times\vec{A})$ in Polar $H_\phi = \frac{1}{\mu}(\nabla\times A_\phi) = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$

From our prev derivative where A_r, A_θ & A_ϕ are related with A_x, A_y & A_z we got $A_x = 0 = A_y$ $A_r = A_z\cos\theta$, $A_\theta = -A_z\sin\theta$, $A_\phi = 0$.

For $A_r = A_z\cos\theta = 0 \Rightarrow$ current is entirely along z direction.

$$H_\phi = \frac{1}{\mu}\left[\frac{1}{r}\frac{\partial(r(-A_z\sin\theta))}{\partial r}\right]$$

$$= \frac{1}{\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{-r\times\frac{\mu I_m e^{-j\beta r}}{2\pi r\beta}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right]\sin\theta\right\}\right]$$

$$= \frac{-I_m}{2\pi r\beta}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]\frac{\partial(e^{-j\beta r})}{\partial r} = \frac{-I_m}{2\pi r\beta}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]\left[-j\beta e^{-j\beta r}\right]$$

$$= \frac{jI_m e^{-j\beta r}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$ ⇒ magnitude of magnetic field strength of Radiation pattern of a quarter wave monopole or Half wave Dipole.

$$|H_\phi| = \frac{I_m}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

Again $E_\theta = \eta H_\phi$ ∴ $E_\theta = \frac{j\eta I_m e^{-j\beta r}}{2\pi r}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]$ Putting $\eta = 120\pi$

$$E_\theta = \frac{j60\pi I_m e^{-j\beta r}}{r}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right] \qquad |E_\theta| = \frac{60 I_m}{r}\frac{\cos(\pi/2\cos\theta)}{\sin\theta}$$

Average Power radiated using Poynting vector theorem

$$P_{avg} = \frac{1}{2}\left\{|E_\theta|\,|H_\phi|\right\} = \frac{1}{2}\left\{\frac{60 I_m}{r}\left[\frac{\cos(\pi/2 \cos\theta)}{\sin\theta}\right] \times \frac{I_m}{2\pi r}\left[\frac{\cos(\pi/2 \cos\theta)}{\sin\theta}\right]\right\}$$

$$= \frac{15 I_m^2}{\pi r^2}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2 = \frac{30 I_{rms}^2}{\pi r^2}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2$$

Total Power radiated by a $\lambda/4$ monopole can be calculated by integrating average power over half shere.

$$P_{total}\,(\text{monopole}) = \oint_s P_{avg}\, dA = \int_0^{2\pi}\int_0^{\pi/2} P_{avg}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore P_{tot}(m) = \int_0^{2\pi}\int_0^{\pi/2} \frac{30 I_{rms}^2}{\pi \cancel{r^2}}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2 \cancel{r^2} \sin\theta\, d\theta\, d\phi$$

$$= \frac{30 I_{rms}^2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{30 I_{rms}^2}{\pi}(2\pi)\int_0^{\pi/2}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2 \sin\theta\, d\theta = 60 I_{rms}^2 \int_0^{\pi/2}\left[\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right]^2 \sin\theta\, d\theta$$

$$= 60 I_{rms}^2 \int_0^{\pi/2} \frac{\left[\cos(\pi/2\cos\theta)\right]^2}{\sin\theta}\, d\theta$$ Solve this Numerically. (How) →

we use here Analytical Process :-

$$\int_0^{\pi/2}\frac{\cos^2(\pi/2\cos\theta)}{\sin\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{1+\cos(\pi\cos\theta)}{\sin\theta}\,d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$ $\therefore \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

Let $u = \cos\theta$ $\therefore du = -\sin\theta\, d\theta$ $\therefore \frac{d\theta}{\sin\theta} = -\frac{du}{\sin^2\theta} = -\frac{du}{1-u^2}$

$\sin^2\theta + \cos^2\theta = 1$ $\sin^2\theta = 1 - u^2$

$$\therefore \int_0^{\pi/2}\frac{\cos^2(\pi/2\cos\theta)}{\sin\theta}\,d\theta = -\frac{1}{2}\int_1^0\frac{(1+\cos\pi u)}{1-u^2}\,du = -\frac{1}{4}\int_0^1(1+\cos\pi u)\left[\frac{1}{1+u}+\frac{1}{1-u}\right]du$$

$$= \frac{1}{4}\int_{-1}^{1}\frac{1+\cos\pi u}{1+u}\,du$$

Let $v = \pi(1+u)$ $\therefore dv = \pi\, du$ $\therefore \frac{dv}{v} = \frac{du}{1+u}$

$\therefore \pi u = v - \pi$

Now $\cos\pi u = \cos(v-\pi) = \cos v\cos\pi + \sin v\sin\pi = -\cos v$

Therefore $\frac{1}{4}\int_{-1}^{+1}\frac{1+\cos\pi u}{1+u}\,du = \frac{1}{4}\int_0^{2\pi}\frac{1-\cos v}{v}\,dv$

u limit $-1 \to 1$

v limit $0 \to 2\pi$

$$= \frac{1}{4}\int_0^{2\pi}\left(\frac{v}{2!} - \frac{v^3}{4!} + \frac{v^5}{6!} - \frac{v^7}{8!} + \cdots\right)dv = \frac{1}{4}\left(\frac{v^2}{2\cdot 2!} - \frac{v^4}{4\cdot 4!} + \frac{v^6}{6\cdot 6!} - \frac{v^8}{8\cdot 8!} + \cdots\right)_0^{2\pi}$$

v upper limit $2\pi = 6.2832$ & $2\cdot 2! = 4$ | $\frac{v^2}{2\cdot 2!} = 9.87$

Putting all values $\int_0^{\pi/2}\frac{\cos^2(\pi/2\cos\theta)}{\sin\theta}\,d\theta = 0.6093$

Hence $P_t(m) = 60 I_{rms}^2 \times 0.609 = (36.54)\, I_{rms}^2$

comparing with Ant eqv ckt concept Radiated Power vs $P_t(m) \Rightarrow 36.5\,\Omega$ is the radiation Resistance of $\lambda/4$ monopole.

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Similarly P_t (Dipole) $= \int_0^{2\pi}\int_0^{\pi} P_{avg}\, r^2 \sin\theta \, d\theta \, d\phi$

$= 60 I_{rms}^2 \int_0^{\pi} \frac{\cos^2(\pi/2 \cos\theta)}{\sin\theta} d\theta = 60 I_{rms}^2 \times 1.218 \rightarrow$ Similarly obtained.

$= 73.08\, I_{rms}^2$ → R_r for Dipole.

Power Radiated from halfwave Dipole is twice the Power Radiated from quarter wave monopole. And hence the resistance is twice.

$\boxed{\text{* Question: How much imaginary part is there in Dipole radiation resistance}}$??

Prev Page one integral was performed

$\int_0^x \frac{1-\cos v}{v} dv$ it is designated as $S_i(x)$. This is called cosine integral of $x \rightarrow$ ~~Si(x) =~~ $C_i(x) = -\int_x^{\infty} \frac{\cos v}{v} dv$

and

Sine integral $S_i(x) = \int_0^x \frac{\sin v}{v} dv$

In general. $C_i(x) = \ln x + C - S_i(x)$ and $C = 0.577$

$S_i(x) = \int_0^x \frac{1-\cos v}{v} dv = \left(\frac{x^2}{2 \cdot 2!} - \frac{x^4}{4 \cdot 4!} + \frac{x^6}{6 \cdot 6!} - \cdots\right)$

Key Factors and Computation for Different zone of Measurement :-

We have already got the information about reactive near field, radiating far field and radiating near field. For ease of computation we do perform approximation to find out field values in far field. The solution provide a close form expression using the approximation. The same can be obtained without approximation, which results in higher order terms subject to the antenna aperture dimension.

We know $A(x,y,z) = \frac{\mu}{4\pi}\int_c I(x', y', z') \frac{e^{-jKR}}{R} dL$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ | length of R is defined as the distance from any point on the source to the observation point

For infinitesimal dipole we have approximated by $R = r$. (r is the distance between origin and P)

But for a finite length thin dipole, it is symmetrically positioned about the origin and it's length variation is along z. As the wire is very thin $x' = y' = 0$

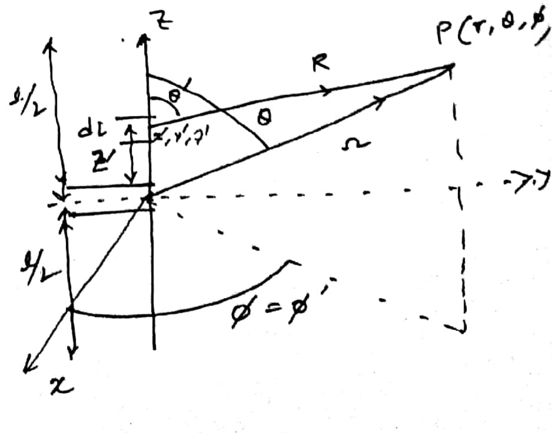
$\therefore R = \sqrt{x^2 + y^2 + (z-z')^2} = \sqrt{(x^2+y^2+z^2) + (-2zz' + z'^2)}$

$= \sqrt{r^2 + (-2z' r\cos\theta) + z'^2}$ | on a sphere r can be represented by x, y, z

using Binomial expansion on

$\sqrt{r^2 + (-2rz'\cos\theta + z'^2)}$

$= r - z'\cos\theta + \underbrace{\frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)} + \underbrace{\frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)} + \cdots$



Now at far field region we can assume a plane wave $R \approx r$

$$R = r - z'\cos\theta$$

Here third term $\frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)_{max} = \frac{z'^2}{2r}$ for $\theta = \pi/2$ | Fourth term at $\theta = \pi/2$ is zero.

So the maximum error we are compromising is $\frac{z'^2}{2r}$.

It has been observed experimentally that $l > \lambda$ a maximum total phase error of $\pi/8$ rad or 22.5° can be included if we ~~neglect~~ introduce the higher order terms.

$$\therefore \quad \underbrace{\frac{k(z')^2}{2r} \leq \pi/8}_{\text{Maximum phase error}} \qquad \Bigg| \quad \frac{e^{-jkR}}{R} \quad \text{Now } kR = \underbrace{kr - kz'\cos\theta}_{\text{this part as usual}} + \underbrace{\frac{kz'^2}{2r}}_{\text{this part provide phase error.}}$$

this term should be neglected.

For z varying $-L/2$ to $L/2$ $\left(z'_{max} = \frac{L}{2}\right)$

$$\frac{2\pi}{\lambda} \cdot \frac{l^2}{4 \times 2r} \leq \pi/8 \Rightarrow \frac{l^2}{4r\lambda} \leq \frac{1}{8} \Rightarrow \boxed{r \geq 2\left(\frac{l^2}{\lambda}\right)}$$

This shows that to maintain the maximum phase error of an antenna equal to or less than $(\pi/8)$ rad, the observation distance r must be equal to or greater than $2l^2/\lambda$ where l is the longest ~~distance~~ dimension of antenna. That is why, we make the approximation and in in dipole calculation that in phase e^{-jkR} replace R by $r - z'\cos\theta$. and in denominator $R = r$. $\boxed{r \geq \frac{2D^2}{\lambda}}$ if D is largest dimension.

If the $r < \frac{2l^2}{\lambda}$ then the maximum phase error by the approximation is greater than 22.5°. This is undesirable in many application. If we choose the observation distance smaller than third term and fourth term both will ~~be~~ present.

$$\therefore \quad R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)$$

Now we neglecting the fourth term. So it is to find out how much error we can introduce due to this approximation. So to neglect we need to find out for which condition of θ what are the values we are neglecting. | To get maxima

$$\therefore \quad \frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\right] = \frac{z'^3}{2r^2}\sin\theta\left[-\sin^2\theta + 2\cos^2\theta\right] = 0$$

$\theta = 0$ is minimum error

Now at $\theta = 0$ this will zero which is not max value.

Max error occurs when the second term vanishes - so only $z'^3/2r^2(\sin\theta)$ remains $\left[-\sin^2\theta + 2\cos^2\theta\right]\Big|_{at\ \theta=\theta_1} = 0 \Rightarrow \theta_1 = \tan^{-1}(\pm\sqrt{2})$

Now we assume that max phase error for fourth term is also equal to $\pi/8$ rad then

$$\frac{kz'^3}{2r^2}\cos\theta\sin^2\theta\Bigg|_{\substack{z'=l/2 \\ \theta=\tan^{-1}\sqrt{2}}} = \frac{\pi}{\lambda} \cdot \frac{l^3}{8r^2}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{3}\right) \leq \pi/8$$

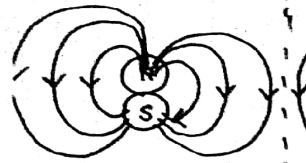
$$\therefore \quad r^2 \geq \frac{2}{3\sqrt{3}}\left(\frac{l^3}{\lambda}\right) = .385\left(\frac{l^3}{\lambda}\right) \quad \Bigg| \quad r \geq 0.62\sqrt{\frac{l^3}{\lambda}}$$

$\therefore$ So ~~the~~ a value of r greater than the ~~ab~~ above value will lead to an error less than $\pi/8$ rad. thus the region where first three ~~term~~ are significant and omission of fourth introduces a phase error of $\pi/8$ is

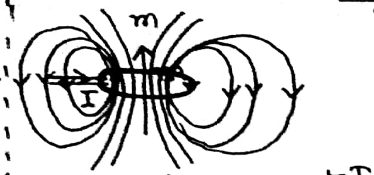
$$\left(2l^2/\lambda\right) > r \geq 0.62\sqrt{l^3/\lambda}$$

[This is called radiating near field

Loop antenna



Bar magnet
Magnetic Dipole



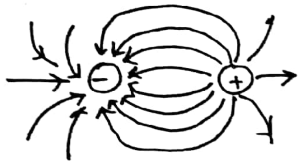
Electric current I
m → Moment
I → loop current

A magnetic Dipole is the closed loop of electric current or a pair of poles as the dimension of source are reduced to zero while keeping identical magnetic moment. It is a magnetic analogue of electric dipole.

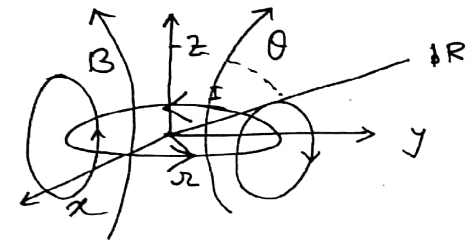
Magnetic field of a dipole is calculated as the limit of either current loop or a pair of charges as source shrinks to a point while keeping magnetic moment 'm' constant.

$$A = \frac{\mu_0}{4\pi r^2} \frac{m \times r}{r}$$

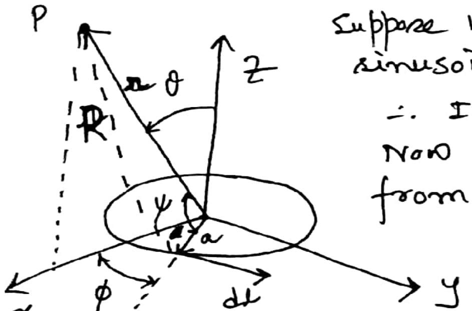
$4\pi r^2$ is the surface of a sphere of radius r.



$$B = \nabla \times A = \frac{\mu_0}{4\pi}\left(\frac{3r(m\cdot r)}{r^5} - \frac{m}{r^3}\right)$$



How a magnetic Dipole Radiate



Suppose we have a wire loop of radius a around which a current sinusoidal in nature is acting at a frequency ω.

$\therefore I(t) = I_0 \cos\omega t$

Now we can imagine this as oscillating magnetic Dipole from the analogy as shown above.

$\therefore$ Magnetic Dipole moment

$$m(t) = \pi a^2 I(t)\,\hat{k} = m_0 \cos\omega t\ \hat{k}$$

where $m_0 = \pi a^2 I_0$ → max value of dipole moment.

Loop does not contain any static charge ⇒ No scalar potential.
So from the concept of retarded vector potential

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{I_0 \cos\omega(t - R/c)}{R}\, dl$$

The circular loop can be assumed of infinite number of infinitesimal current element. The position of current element in xy plane is ~~ch~~ characterized by $0° < \phi < 360°$ and $\theta = 90°$

Now Direction of $\vec{A}$ is in y direction. x component have symmetry
Thus solution of A does not depends on φ.

| ψ is the angle between vector r & a

$$\cos\psi = \cancel{\hat{x}\sin\theta\cos} \ \hat{r}\cdot\hat{a}$$
$$= (\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\cdot(\hat{x}\cos\phi + \hat{y}\sin\phi)$$

For actual case r and a ≠ 1 and φ = 0° for symmetry. (only for r not for a as r is symmetrical in all aspect)

$$\therefore \cos\psi\,(r\cdot a) = (r\sin\theta\ \hat{x} + r\cos\theta\ \hat{z})\cdot(a\,\hat{x}\cos\phi + a\sin\phi\ \hat{y})$$
$$= ra\sin\theta\cos\phi$$

[as other components have dot product between $\hat{x}\hat{y}$, $\hat{z}\hat{x}$ $\hat{z}\hat{y}$]

Now From Law of cosine triangle we know

$$R = \sqrt{r^2 + a^2 - 2ra\cos\psi}$$

Now from symmetry we can assume loop radius constant and current constant for a small section. (43)

$$A(r,t) = \frac{\mu_0}{4\pi} I_0 a \int_0^{2\pi} \frac{\cos\omega(t - R/c)}{R} \cos\phi \, d\phi$$

$dl = a\cos\phi \, d\phi$

$a\cos\phi$ Projection on x axis & $a\cos\phi\, d\phi$ is loop area.

and $R = \sqrt{r^2 + a^2 - 2ra\cos\psi}$

$$= \sqrt{r^2 + a^2 - 2ra\sin\theta\cos\phi}$$

Now we have assume that Magnetic Dipole should be closely spaced. Also Point of observation is far enough $\Rightarrow r \gg a$

$$\therefore R = r\left(1 - \frac{a}{r}\sin\theta\cos\phi\right) \Rightarrow \frac{1}{R} = \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right)$$

$$\therefore \cos\omega(t - R/c) = \cancel{\cos\left[\omega\left(t - \frac{R}{c}\right)\right.} + \cos\omega\left(t - \frac{r\left(1 - \frac{a}{r}\sin\theta\cos\phi\right)}{c}\right)$$

$$= \cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega a}{c}\sin\theta\cos\phi\right]$$

$$= \cos\omega\left(t - \frac{r}{c}\right)\cos\underbrace{\left(\frac{\omega a}{c}\sin\theta\cos\phi\right)}_{} - \sin\omega\left(t - \frac{r}{c}\right)\sin\underbrace{\left(\frac{\omega a}{c}\sin\theta\cos\phi\right)}_{}$$

Also $\omega \gg a \rightarrow a \ll \frac{c}{\omega}$ $\left(\cos x = 1\right)$ $\left(\sin x = x\right)$

$$\therefore \boxed{\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega a}{c}\sin\theta\cos\phi\sin\omega\left(t - \frac{r}{c}\right)}$$

and $\boxed{\frac{1}{R} = \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right)}$

Now Putting the values

$$A(r,t) = \frac{\mu_0}{4\pi} I_0 a \int_0^{2\pi} \frac{\cos\omega(t - R/c)}{R}\cos\phi\, d\phi$$

$$= \frac{\mu_0 I_0 a}{4\pi}\int_0^{2\pi} \left[\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega a}{c}\sin\theta\cos\phi\sin\omega\left(t - \frac{r}{c}\right)\right] \times \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right)\cos\phi\, d\phi$$

$$= \frac{\mu_0 I_0 a}{4\pi r}\int_0^{2\pi}\left\{\underbrace{\cos\omega\left(t - \frac{r}{c}\right)}_{\text{with } \cos\phi} + a\sin\theta\cos\phi\left[\frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right)\right]\right\}\cos\phi\, d\phi$$

First integration $\rightarrow 0$

Now $\int_0^{2\pi}\cos\phi\, d\phi = 0$ from symmetry.

$$\int_0^{2\pi}\cos^2\phi\, d\phi = \pi$$

$$\therefore A = \frac{\mu_0 I_0 a \cdot a}{4\pi r}\int_0^{2\pi}\sin\theta\cos^2\phi\left[\frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right)\right]d\phi \;\hat{\phi}$$

$$= \frac{\mu_0 I_0 a^2\pi}{4\pi}\left(\frac{\sin\theta}{r}\right)\left[\frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right)\right]\hat{\phi}$$

$$= \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left[\frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right)\right]\hat{\phi}$$

Reference Griffith's Book Page no 238 Magnetic Dipole Fig no 5.53

Now If $\omega \Rightarrow 0$ for static $\rightarrow$ $A(r,\theta) = \frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}\hat{\phi}$

Now as $r \gg a$ and $r \gg \frac{c}{\omega}$

$$A = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\omega\left(t-\frac{r}{c}\right)\hat{\phi}$$

First term negligible

$$\therefore E = -\frac{\partial A}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\phi}$$

and $$B = \nabla\times A = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta}$$

Solve $\nabla\times A$ and Put it in B.

See that E and B are $\perp^r$ in nature and both transverse to the direction of Propagation vector $\hat{r}$. Also $E_0/B_0 = c$.

But what is the most important Difference

in Electric Dipole $E\,\hat{\theta}$ and $B\,\hat{\phi}$

in Magnetic Dipole $E\,\hat{\phi}$ and $B\,\hat{\theta}$

See the duality.

$$S = \frac{1}{\mu_0}(E\times B) = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\right]^2\hat{r}$$

and intensity $$\langle S\rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

Total radiated power $$\langle P\rangle = \left(\frac{\mu_0 m_0^2}{12\pi c^3}\right)\omega^4$$

See notes for electric Dipole with retarded Potential

So the same Donut shape and Power radiated goes as fourth power of frequency.

$$\langle S\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r} \qquad \langle S\rangle = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

$$\underbrace{\langle P\rangle = \frac{1}{4\pi\epsilon_0}\frac{p_0^2\omega^4}{3c^3}}_{\text{Electric}} \qquad \underbrace{\langle P\rangle = \frac{\mu_0 m_0^2}{12\pi c^3}\omega^4 = \frac{1}{4\pi\epsilon_0}\frac{m_0^2\omega^4}{3c^5}}_{\text{magnetic}}$$

$$\frac{P_{magnetic}}{P_{electric}} = \text{ratio of Power} = \left(\frac{m_0}{p_0 c}\right)^2$$

$m_0 = \pi a^2 I_0$

$p_0 = q_0 s$

If $I_0 = q_0\omega$ and $s = \pi a$ (assume)

$$= \left(\frac{a\omega}{c}\right)^2$$

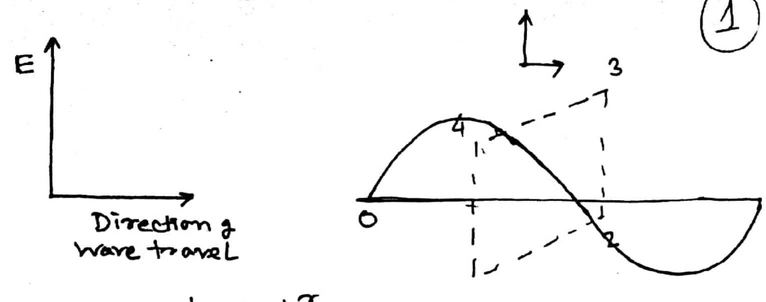
$\therefore$ $\boxed{\text{Electric Dipole Dominates over Magnetic Dipole}}$

Application of loop antenna

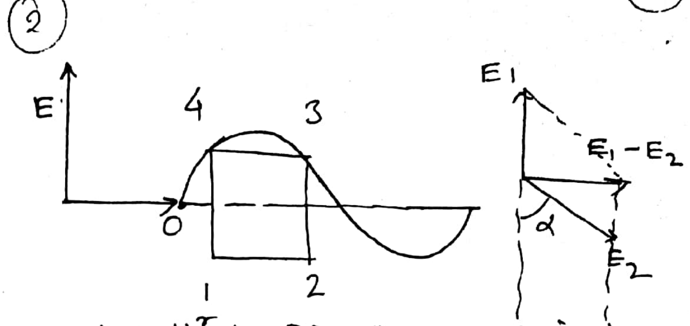
Loop antenna can be of different shape — circular, rectangular, square, triangular. Mostly use as circular or square loop. It is a radiating coil of any cross section and can have one or more turn for carrying RF current.

It is mostly used for radio direction finding. Radio direction finding refers to a method of direction finding of unknown transmitter based on received signal strength. Directional property of any loop depends upon the phase between the turns and this can be enhanced by proper phasing. In general there are two kinds of loop → small loop & large loop. The circumference of small loop antenna is smaller than one wavelength ($A<\lambda$) while for Large loop $A \geq \lambda$.

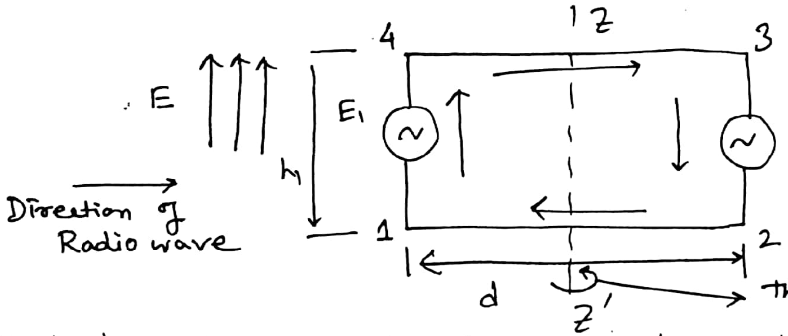
Direction Finding by loop antenna :-



Loop ⊥ to Direction of arrival



loop ∥ to Direction of arrival



This is the rotation shown in above fig.

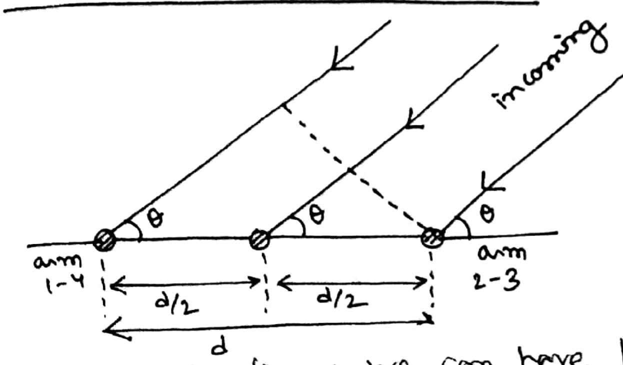
For a single turn loop of rectangular shape four vertices are numbered as 1, 2, 3, 4. The side 1–4 and 2–3 are vertical arms while 1–2 & 3–4 are horizontal arms. The antenna can be rotated around its axis Z-Z' and it passes through middle of the loop.

Now in First (1) configuration, the plane of antenna is 90° to direction of incoming wave. Hence all the arms are equidistant from transmitter. If the incoming wave is vertically polarized then induced e.m.f in arm 1–4 and 2–3 are of same amplitude. But there will be a close loop current flow in loop. And hence current in 1–4 & 2–3 arms are same magnitude but opposite phase. Hence induced e.m.f in vertical arms get cancelled. No emf in horizontal arms.

Now for second case (2) the loop is in ∥ to direction of wave. In this case the emf induced in arm 1–4 and 2–3 have same magnitude but with different phase due to extra distance d to be travelled by wave. Thus the resultant emf at ZZ' is $E_1 - E_2$. The maximum value of the ~~loop~~ emf is achieved when the plane of loop is in exact direction with incoming wave.

If θ be the angle between incoming wave and plane of loop then induced emf in vertical axis is expressed as $E_\theta = E_{rms} \cos\theta$. Thus for $\theta = 0°$ or $180°$ E_{max} and for $\theta = 90°$ or $270°$ E_{min}.

Induced EMF in loop antenna



From the previous concept we saw that E_1 & E_2 are not in phase but has an $\alpha \rightarrow$ phase difference. This α is introduced due to additional distance d/2 travelled by incoming wave with respect to the center of the loop. We can transfer this phase difference by path difference $\Rightarrow \alpha = \beta L$

$$\alpha = \frac{2\pi}{\lambda} \times \frac{d}{2} \cos\theta$$

. The loop has rectangular cross section with h is height and d is width.

So from the figure we can have phase lead and lag by α relative to center. So w.r.t to center the difference between induced emf can be written as

$$E_m h \sin(\omega t + \alpha) - E_m h \sin(\omega t - \alpha) = E_m h \left[\sin(\ ) - \sin(\ )\right]$$

$$E(\theta) = E_m h \sin(\omega t + \alpha) - E_m h \sin(\omega t - \alpha)$$

$$= 2 E_m h (\sin\alpha) \cos\omega t$$

For a farfield case α is very small angle and as $\sin\theta \approx \theta$ for θ small

$$E(\theta) = 2 E_m h \alpha \cos\omega t$$

Putting $\alpha \rightarrow$ $E(\theta) = \dfrac{2E_m \cancel{2}\pi h d}{\lambda \times \cancel{2}} \cos\omega t \cos\theta = \dfrac{2\pi h d \cos\theta}{\lambda} E_m \cos\omega t$

If the loop has N no of turns and cross section $A = hd$ then

$$E(\theta) = \frac{2\pi N A \cos\theta}{\lambda} E_m \sin\left(\pi/2 + \omega t\right) = V_m \sin\left(\frac{\pi}{2} + \omega t\right)$$

where $\boxed{V_m = \frac{2\pi N A \cos\theta}{\lambda} E_m}$ & $\boxed{V_{rms} = \frac{2\pi N A \cos\theta}{\lambda} E_{rms}}$

$\boxed{\frac{2\pi NA}{\lambda} = \text{Effective height of loop}}$

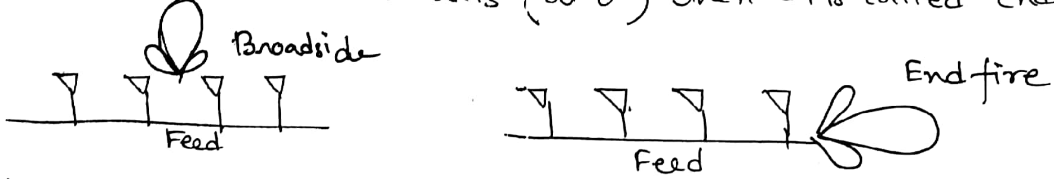
Thus it defines that rms value of induced emf at center of loop which is a function of angle between incoming wave and plane of loop.

E_{max} will occur for $\theta = 0°$.

Antenna Array

An antenna array is a combination of many antennas separated from each other to obtain the desired radiation characteristics.. Each antenna in antenna array is termed as element. The overall radiation characteristics of antenna array depends upon the element used and the spacing between elements.

Antenna array mainly divided into two types, namely broadside array and end fire array. If the beam of the antenna is formed at normal to the antenna (at 90°) ~~to the ant~~ then it is defined as Broadside array. If the antenna array beam is in line with antenna axis (at 0°) then it is called end fire array.



There are other two special type of antenna array $\longrightarrow$ Parasitic Array and Collinear array.

Point source and Array $\Rightarrow$ Finding out Array Factor (Array Analysis)

To predict antenna array and their characteristics the simplest way is to assume antenna element as point source. Thus every source is a hypothetical radiator with zero volume.

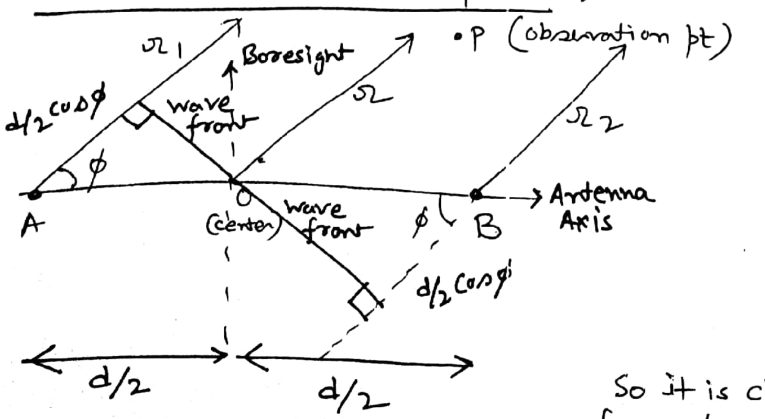
our calculation starts with two point source

(A) Two point sources fed by equal currents in same phase

(B) Two point sources fed by equal currents but opposite phase

(C) Two point sources fed by unequal current in any phase

A) Equal current same phase :-



Consider an array of two point sources say A & B. They distance d apart and d/2 from origin O. Both the point sources are fed with equal current & same phase. Observation point is P which is at a distance r_1 & r_2 for source A & B respectively. r_1 & r_2 are very large compared to d. Each source radiates at an angle ϕ relative to the antenna axis.

So it is clear from wavefront that wave coming from A will reach later as compared to O.

Hence the ~~phat~~ path difference between OP & AP can be defined as

~~Phat~~ Path Difference $= r - r_1 = -\frac{d}{2}\cos\phi$. So phase difference $\psi_1 = \beta\left(-\frac{d}{2}\cos\phi\right)$

$$= -\frac{2\pi}{\lambda}\times\frac{d}{2}\cos\phi = -\frac{\pi d}{\lambda}\cos\phi$$

Similarly wave coming from B will reach earlier to P compared to O. Hence path difference between the path OP & BP can be defined as

Path Diff $= r - r_2 = \frac{d}{2}\cos\phi \Rightarrow \psi_2 = \frac{2\pi}{\lambda}\cdot\frac{d}{2}\cos\phi = \frac{\pi d}{\lambda}\cos\phi$

Therefore the phase difference between B and A can be calculated as

$$\psi = \psi_2 - \psi_1 = \frac{\pi d}{\lambda}\cos\phi + \frac{\pi d}{\lambda}\cos\phi = \frac{2\pi d}{\lambda}\cos\phi = \beta d\cos\phi$$

By comparing ψ with ψ_1 & ψ_2 we can write that $\psi_1 = \frac{-\psi}{2}$ & $\psi_2 = \frac{\psi}{2}$

It means phase of wave coming from source A is lagging relative to boresight as it travels more distance by $\frac{d}{2}\cos\phi$. Similarly wave coming from B is leading relative to origin

Now we need to find out total electric field at far field point P due to individual field component of A & B. If the magnitude of field are same then for $|E_1| = |E_2| = E_0$ we can write

$$E_1 = E_0 e^{-j\psi/2} \quad \text{and} \quad E_2 = E_0 e^{j\psi/2}$$

$$\therefore E = E_1 + E_2 = E_0 e^{-j\frac{\psi}{2}} + E_0 e^{j\frac{\psi}{2}} = E_0\left[e^{-j\frac{\psi}{2}} + e^{j\frac{\psi}{2}}\right] = 2E_0\cos\frac{\psi}{2}$$

Putting $\psi \rightarrow$ we get $\boxed{E = 2E_0\cos\left(\frac{\beta d\cos\phi}{2}\right)}$ At pt P total field due to two pt source having excited by same current and same phase.

Mag of $E_{max} = 2E_0$ & $\angle E = \frac{\beta d\cos\phi}{2}$

Normalized Field strength at P $= E_{norm} = \frac{|E|}{|E_{max}|} = \frac{\left|2E_0\cos\left(\frac{\beta d\cos\phi}{2}\right)\right|}{|2E_0|}$

$\therefore \boxed{E_{norm} = \left|\cos\left(\frac{\beta d\cos\phi}{2}\right)\right|}$ → This is known as Array Factor which is Normalized.

Maxima :- From normalized Array Factor we get maximum value of far field can be achieved if $E_{norm} = 1$.

$$\therefore \left|\cos\left(\frac{\beta d\cos\phi}{2}\right)\right| = 1 \rightarrow \cos\left(\frac{\beta d\cos\phi}{2}\right) = \pm 1$$

Now if we assume $d = \lambda/2$ (half wave length spacing) then

$$\cos\left(\frac{\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\cos\phi}{2}\right) = \pm 1 \Rightarrow \cos\left(\frac{\pi\cos\phi}{2}\right) = \pm 1 \Rightarrow \frac{\pi\cos\phi}{2} = \cos^{-1}(\pm 1)$$

$$\therefore \frac{\pi\cos\phi}{2} = \pm n\pi \quad \{n = 0, 1, 2 \ldots\}$$

For $n = 0 \Rightarrow \left(\frac{\pi\cos\phi_{max}}{2}\right) = 0 \Rightarrow \phi_{max} = 90^\circ$ or 270°

It suggests that maximum far field is achieved in the direction of Boresight or $\perp^r$ to the array axis. Thus it gives that array of two point sources with equal current in same phase ~~gi~~ is a BROADSIDE ARRAY.

Minima :- To get minima $E_{norm} = 0 \Rightarrow \cos\left(\frac{\beta d \cos\phi}{2}\right) = 0$

For $d = \lambda/2 \rightarrow \cos\left(\frac{\pi\cos\phi}{2}\right) = 0 \Rightarrow \left(\frac{\pi\cos\phi}{2}\right) = \cos^{-1}(0) = \pm\frac{(2n+1)\pi}{2}$

For $n=0$ $\left(\frac{\pi\cos\phi_{min}}{2}\right) = \pm\pi/2 \Rightarrow \cos\phi_{min} = \pm 1 \Rightarrow \phi_{min} = 0^\circ$ or 180°

So minima of farfield appears at the array axis.

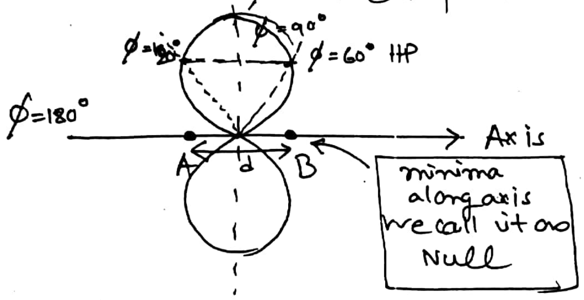
Half Power Point :- At half Power Point $E_{norm} = \frac{1}{\sqrt{2}}$

$\therefore \cos\left(\frac{\beta d\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}$ for $d = \frac{\lambda}{2}$ $\cos\left(\frac{\pi\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}} \Rightarrow \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$

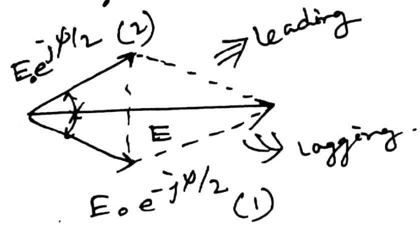
$\therefore \frac{\pi\cos\phi}{2} = \pm\frac{(2n+1)\pi}{4}$ | For $n=0$ $\frac{\pi\cos\phi_{HP}}{2} = \pm\pi/4 \Rightarrow \phi_{HP} = 60^\circ$ or 120°

HP point at Far field appears at 60° from array axis.

Field Patterns :- Field pattern represent the variation of total (E) with respect to ϕ. Field pattern will look like dumble shape



Phasor Diagram



3) **Equal Current and opposite Phase Excitation** !-

~~For the Som~~ For the same figure, if the magnitude of input current remain same but current phase are opposite for point source A and B. (phases of input current fed to source A and B should be equal to 180°). Hence the Farfield due to point source A and B can be written as

$E_1 = E_0 e^{j\pi} = E_0(\cos\pi + j\sin\pi) = -E_0$

and $E_2 = E_0 e^{j0} = E_0$

[Please note this e^{jx} is not identical with $e^{j\psi}$ as this e^{jx} is related with E field only]

So again we know $\psi_1 = -\psi/2$ and $\psi_2 = \psi/2$

$\therefore E = E_1 + E_2 = -E_0 e^{-j\psi/2} + E_0 e^{j\psi/2} = 2jE_0\left[\frac{e^{j\psi/2} - e^{-j\psi/2}}{2j}\right] = 2jE_0\sin\frac{\psi}{2}$

Again we know $\psi = \beta d\cos\phi$ and $\psi/2 = \frac{\beta d\cos\phi}{2}$

$\therefore \psi/2 = \pi/2\cos\phi \Rightarrow E = 2jE_0\sin(\pi/2\cos\phi)$

Maxima :- Normalized Far field strength is given by $E_{norm} = \frac{|E|}{|E_{max}|} = \frac{|\cancel{2j E_0}\sin(\pi/2\cos\phi)|}{|\cancel{2E_0}|}$

$\therefore E_{norm} = |\cancel{2j E_0}\sin(\pi/2\cos\phi)| \rightarrow$ The term j represents ~~the~~ only the phase component.

So taking only magnitude $E_{norm} = |\sin(\pi/2\cos\phi)|$

Now for maxima $E_{norm} = 1$ so substituting the value $|\sin(\pi/2\cos\phi)| = 1$

$\sin(\pi/2\cos\phi) = \pm 1 \Rightarrow \pi/2\cos\phi = \sin^{-1}(\pm 1) = \pm\frac{(2n+1)\pi}{2}$ | For $n=0$ $\cos\phi_{max} = \pm 1$

$\phi_{max} = 0^\circ$ or 180°

Maxima occurs along the antenna axis.

Minima :- For $E_{norm} = 0 \Rightarrow \left|\sin\left(\frac{\pi}{2}\cos\phi\right)\right| = 0 \Rightarrow \frac{\pi}{2}\cos\phi = \sin^{-1}(0) = \pm n\pi$

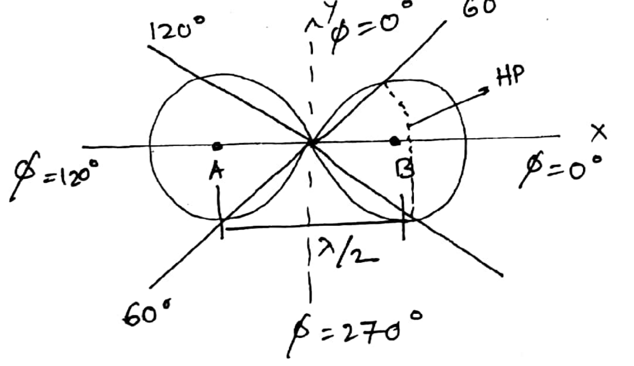
For $n = 0$ $\cos\phi_{min} = 0 \Rightarrow \phi_{min} = 90°$ or $270°$. It means that minima will be obtained along boresight.

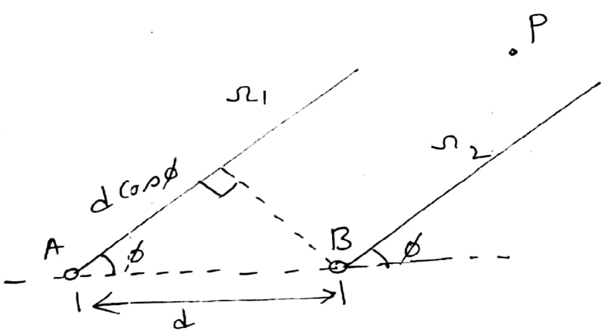
Half Power :- For $E_{norm} = \frac{1}{\sqrt{2}}$ $\quad \sin\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}$

For $d = \lambda/2$ and $\beta = 2\pi/\lambda$ $\quad \frac{\pi}{2}\cos\phi = \sin^{-1}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{(2n+1)\pi}{4}$

For $n = 0$ $\quad \frac{\pi\cos\phi_{HP}}{2} = \pm\pi/4 \Rightarrow \cos\phi_{HP} = \pm\frac{1}{2} \Rightarrow \phi_{HP} = 60°$ or $120°$

Field pattern :





c) Unequal current and any phase :-

Let us assume there are two point sources say A and B separated by distance d. The far field at point P due to source A and B are E_1 & E_2 respectively. If the phase difference between excitation is α and the path difference is $d\cos\phi$ then total phase difference between source A and B can be written as $r_1 - r_2 = \beta d\cos\phi$

Total ~~Path diff~~ phase difference including excitation phase difference can be written as $\quad \psi = \beta d\cos\phi + \alpha$

If point source A is taken as reference then total far field at point P

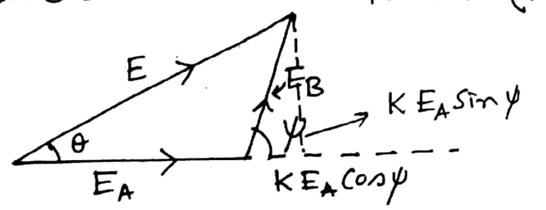
$$E = E_1 + E_2 = E_A e^{j0} + E_B e^{j\psi} = E_A\left(1 + \frac{E_B}{E_A}e^{j\psi}\right) = E_A\left(1 + Ke^{j\psi}\right)$$

K = relative amplitude ratio.

$$\therefore \; E = E_A\left(1 + K\cos\psi + jK\sin\psi\right)$$

$$\Rightarrow |E| = E_A\sqrt{(1+K\cos\psi)^2 + (K\sin\psi)^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{K\sin\psi}{1+K\cos\psi}$$

Resultant Field at Point P (in terms of phasor)



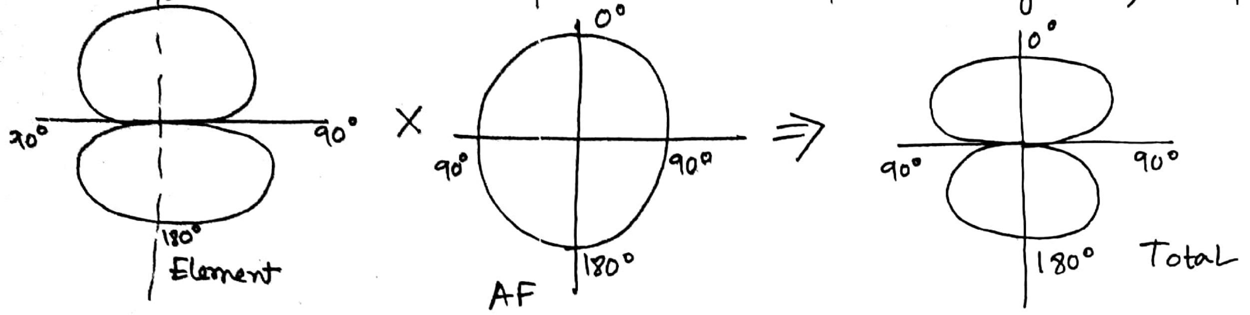
From the two element array we get E_{total} = E (single element) × AF

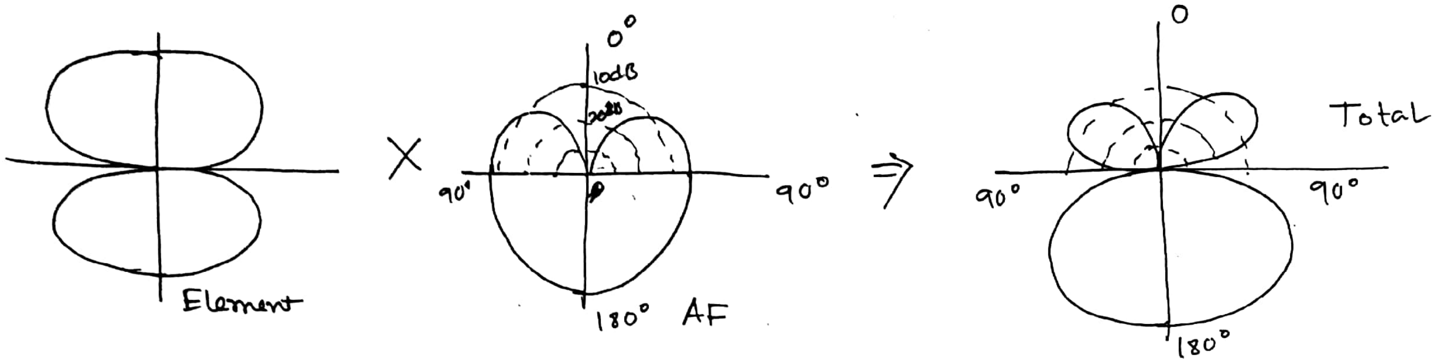
This is referred as Pattern Multiplication for arrays of identical element.

* Array Factor is function of number of element, geometrical arrangement, their relative magnitude, their relative phases and their spacing.

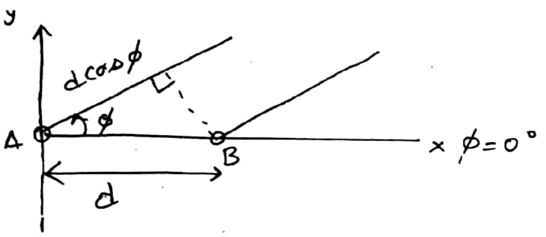
Some example of Pattern multiplication

Refer Balanis Example 6·1 Fig 6·3, Example 6·2 Fig 6·4



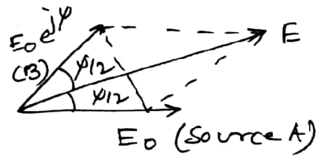


If the Axis of origin is not at center symmetrical as in case 1



Compare case 1 → Distance between two source ⇒ d
But $d/2$ is the distance between origin and source
Here Distance between origin & source A = 0
" " " " & source B = d

Phasor Diagram



Here the reference will start from origin which is also the source A. Path Difference $d\cos\phi$

So $\psi_2 = \beta d\cos\phi$ & $\psi_1 = 0$

Total field $= E = E_0 + E_0 e^{j\psi} = E_0 + E_0 e^{j\beta d\cos\phi}$

$\Rightarrow |E| = 2E_0 \cos\psi/2 = 2E_0 \cos\left(\frac{\beta d\cos\phi}{2}\right)$ ⇒ Magnitude part

$E = E_0\left(1+e^{j\psi}\right) = 2E_0 e^{j\psi/2}\left(\frac{e^{j\psi/2}+e^{-j\psi/2}}{2}\right) = 2E_0 e^{j\psi/2}\cos\psi/2$

So phase of the total field is $\psi/2$

So Normalizing we can write $E = \cos\psi/2 \;\angle\psi/2$

Here the cosine factor gives the amplitude variation of E and the exponential or angle factor gives phase variation. With respect to source 1.

Compare with case 1 $\quad |E| = 2E_0$ & $\angle E = \cos\psi/2$

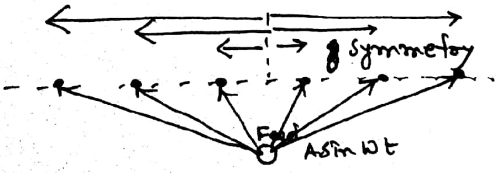
Here $\quad |E| = 2E_0\cos\psi/2$ and $\angle E = \psi/2$

★ Very much careful when you design an array. choose your origin which will give you suitable phase variation.

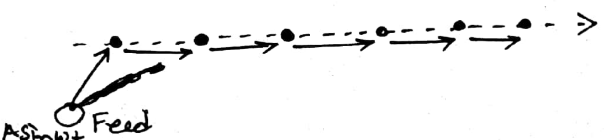
★ Also Refer to Kraus Book Pg-121 chapter 4 Fig 4.2.

Now on the basis of two Difference Design process for same case-1 Practical Design are shown below

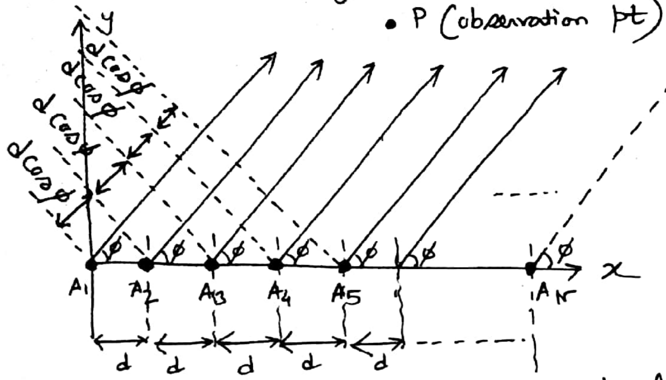
Center Fed



End Fed or Edge Fed.



Uniform linear Array of N point sources (Edge or End Fed)



- linear array all element in straight line.
- Uniform distribution of Power → all the point source fed with equal amount of magnitude of current.
- Spacing between point source d
- N no of point source
- Origin is at A_1, so all the data relative to A_1

Now phase difference to be calculated →

ψ_1 = Phase between A_1 & A_2 = $\frac{2\pi}{\lambda} \times d\cos\phi = \psi$

ψ_2 = Phase difference between A_1 & A_3 = $\beta \times 2d\cos\phi = \frac{4\pi}{\lambda}\cos\phi = 2\psi$

ψ_3 = " " " A_1 & A_4 = $\beta \times 3d\cos\phi = \frac{6\pi}{\lambda}\cos\phi = 3\psi$

ψ_{N-1} = " " " A_1 & A_N = $\beta \times (2N-1)\, d\cos\phi = \frac{2(N-1)\, d\cos\phi}{\lambda} = (N-1)\psi$

As all one having same magnitude exitation hence $|E_1| = |E_2| = |E_3| = \cdots |E_N| = E_0$

So total Field strength at P due to N source is

$$E_t = E_1 e^{j0} + E_2 e^{j\psi_1} + E_3 e^{j\psi_2} + \cdots E_N e^{j\psi_{N-1}}$$

$$= E_0\left[1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots e^{j(N-1)\psi}\right]$$ For all E_0

Now $1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots e^{j(N-1)\psi} = \left[\frac{1-e^{jN\psi}}{1-e^{j\psi}}\right]$

$$\therefore E_t = E_0\left[\frac{1-e^{jN\psi}}{1-e^{j\psi}}\right] = E_0\left[\frac{e^{-j\frac{N\psi}{2}} - e^{j\frac{N\psi}{2}}}{e^{-j\psi/2} - e^{j\psi/2}}\right]\left[\frac{e^{jN\psi/2}}{e^{j\psi/2}}\right]$$

Now $e^{jN\psi/2} - e^{-jN\psi/2} = 2j\sin N\psi/2$ & $e^{j\psi/2} - e^{-j\psi/2} = 2j\sin\psi/2$

taking (−) common in [ ] and substituting exponential terms by imaginary sin part

$$\boxed{E_t = E_0\left[\frac{2j\sin\frac{N\psi}{2}}{2j\sin\psi/2}\right] e^{j(N-1)\frac{\psi}{2}}}$$

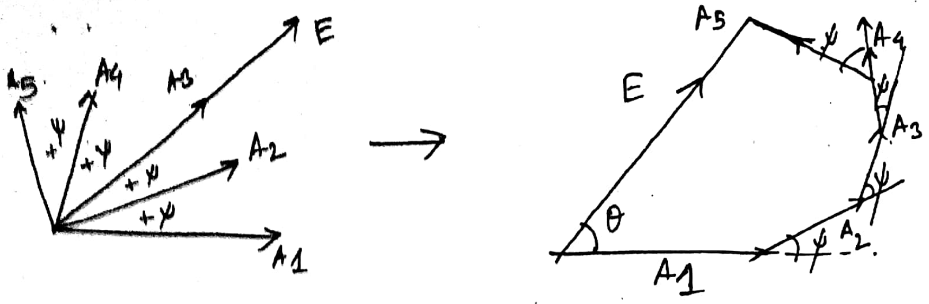
So now the magnitude part is $|E_t| = E_0\left[\frac{\sin N\frac{\psi}{2}}{\sin\frac{\psi}{2}}\right] = E_0\left[\frac{\sin N\frac{(\beta d\cos\phi)}{2}}{\sin\frac{(\beta d\cos\phi)}{2}}\right]$

and $\theta = (N-1)\frac{\psi}{2} = \frac{(N-1)\beta d\cos\phi}{2}$

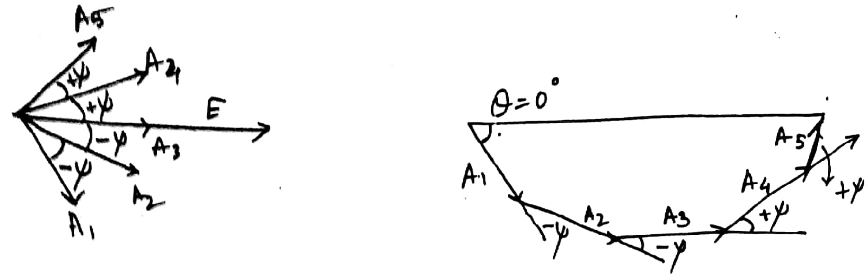
★ Here we assume that current fed to point sources are of same magnitude and same phase. Here the phase variation is coming due to path difference. But now if current sources also have a progressive phase difference of α fed individually then magnitude of E_t remains same but θ of E_t will change to

$$\theta = (N-1)\frac{\psi}{2} = \frac{(N-1)\left[\beta d\cos\phi + \alpha\right]}{2}$$

So if α = 0 we get N element uniform broadside array and if α ± βd we get N element uniform end fire array.



→ for Edge Fed



Now we got the relationship of E_T as $E_T = E_0 \left[\dfrac{\sin N\left(\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \right] e^{j\frac{(N-1)\psi}{2}}$

From this $|E_T| = E_0 \left[\dfrac{\sin \frac{N\psi}{2}}{\sin \frac{\psi}{2}} \right]$ and $\theta = \dfrac{(N-1)\psi}{2} = \dfrac{(N-1)\beta d\cos\phi}{2}$

So the normalized value at far field $E_{norm} = \dfrac{E_T}{|E_T|} = e^{j\frac{(N-1)\psi}{2}}$

Now Major lobe will appear when $E_{T\,norm} = 1$

$\therefore \; e^{j(N-1)\frac{\psi}{2}} = 1 = \cos(N-1)\frac{\psi}{2} + j\sin(N-1)\frac{\psi}{2}$

Comparing real & imaginary part $\cos(N-1)\frac{\psi}{2} = 1$ & $\sin(N-1)\frac{\psi}{2} = 0$

$\Rightarrow (N-1)\frac{\psi}{2} = \cos^{-1}(1) = \pm n\pi \Rightarrow$ For $n=0$ we get $(N-1)\frac{\psi}{2} = 0 \Rightarrow \psi = 0$

$\beta d\cos\phi_{max} = 0 \Rightarrow \phi_{max} = 90°$ or $270°$ [Maxima at Broadside Direction]

Magnitude of major lobe :- $|E_T| = E_0 \left[\dfrac{\sin \frac{N\psi}{2}}{\sin\frac{\psi}{2}} \right]$ at $\psi = 0$ major lobe occur

But if you put $\psi = 0$ then it becomes a zero/zero solⁿ.

So $|E_{T\,max}| = E_0 \lim_{\psi\to 0} \left[\dfrac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}} \right] \Rightarrow$ using L' Hospital rule $= E_0 \lim_{\psi\to 0} \left[\dfrac{\frac{d\sin\frac{N\psi}{2}}{d\psi}}{\frac{d\sin\frac{\psi}{2}}{d\psi}} \right]$

$\Rightarrow E_0 \lim_{\psi\to 0} \left[\dfrac{\left(\cos\frac{N\psi}{2}\right)\left(\frac{N\psi}{2}\right)}{\left(\cos\frac{\psi}{2}\right)\left(\psi/2\right)} \right] = E_0 N$

So the resultant field at point P due to N element array is N times the field strength due to single element.

Direction of Null :-

Nulls are that point when $|E_T| = 0 \Rightarrow E_0 \left| \dfrac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}} \right| = 0 \Rightarrow \sin\frac{N\psi}{2} = 0$

$\frac{N\psi}{2} = \sin^{-1}(0) \Rightarrow \pm m\pi \quad \therefore \frac{N}{\cancel{2}} \times \frac{\cancel{2}\cancel{\pi}}{\lambda} \times d\cos\phi_{min} = \pm m\cancel{\pi} \Rightarrow \dfrac{Nd\cos\phi_{min}}{\lambda} = \pm m$

$\therefore \phi_{min} = \cos^{-1}\left(\dfrac{\pm m\lambda}{Nd}\right)$

Direction of Side Lobe

Side lobes are subsidiary maxima. It means other than major lobe or maxima other maxima are those points where there are no nulls.

$\therefore \sin \frac{N\psi}{2} = 0 \rightarrow$ gives Null similarly $\sin \frac{N\psi}{2} = \pm 1$ gives other pt where there are no nulls but some $|E_T|$

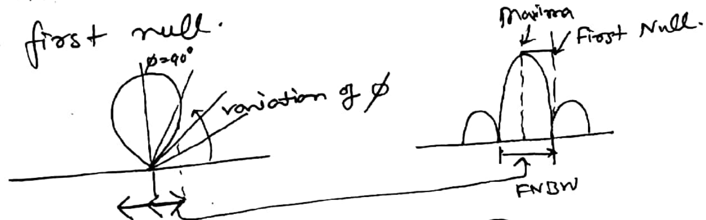
$\therefore \sin \frac{N\psi}{2} = \sin \left[\frac{\pm (2m+1)\pi}{2} \right] \Rightarrow \frac{N\psi}{2} = \frac{\pm(2m+1)\pi}{2}$

Now For $m=0$ $\frac{N\psi}{2} = \pm \frac{\pi}{2} \rightarrow$ gives the principle maxima or major lobe. So we can avoid this value as side lobe.

Now $\psi = \pm \frac{(2m+1)\pi}{N}$ for $m = 1, 2, 3 \ldots$ $[m \neq 0]$

$\frac{2\pi d \cos\phi}{\lambda} = \pm \frac{(2m+1)\pi}{N} \Rightarrow \phi = \cos^{-1}\left[\pm \frac{\lambda(2m+1)}{2Nd}\right]$

Beamwidth :- Beamwidth or FNBW & HPBW. FNBW is the angular distance between two nulls (first). It can also be written as twice the angular difference between the maxima of major lobe and first null.



$\therefore$ FNBW $= 2 \times \delta = 2 \times (90° - \phi) \Rightarrow$

$\therefore \delta = (90° - \phi_{min})$

as Nulls will appear at ϕ_{min}

$\phi_{min} = (90° - \delta) \Rightarrow \cos\phi_{min} = \cos(90° - \delta) = \sin\delta$

Now From Null we got $\phi_{min} = \cos^{-1}\left(\frac{\pm m\lambda}{Nd}\right)$

$\Rightarrow \frac{\pm m\lambda}{Nd} = \cos\phi_{min} = \sin\delta$ | Now for very small δ $\sin\delta = \delta$

$\therefore \delta = \pm \frac{m\lambda}{Nd}$ for first Null $m = 1 \Rightarrow \delta = \left(\frac{\lambda}{Nd}\right)$

Hence FNBW $= 2\delta = \frac{2\lambda}{Nd}$. We can approximate Nd as array length L

$\therefore$ FNBW $= \frac{2\lambda}{L} = 2/(L/\lambda)$ radian $= \frac{114.6}{L/\lambda}$ (degree)

$\therefore$ HPBW $= \frac{\text{FNBW}}{2} = \frac{1}{(L/\lambda)}$ radian $= \frac{57.3}{(L/\lambda)}$ (degree)

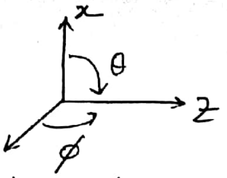
Directivity :- Directivity $D = \frac{U_{max}}{U_{avg}} = \frac{U_{max}}{U_0}$

Avg radiation intensity is defined as ratio of power radiated over the solid angle of 4π $U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |E(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi$

$$|E(\theta,\phi)| = \frac{|E_T|}{|E_{T_{max}}|} = \frac{E_0 \left[\sin \frac{N\psi}{2}\right] \Big/ \left[\sin \psi/2\right]}{NE_0} = \frac{1}{N}\left[\frac{\sin \frac{N\psi}{2}}{\sin \psi/2}\right] = \frac{1}{N}\left[\frac{\sin \frac{N\beta d\cos\phi}{2}}{\sin \frac{\beta d \cos\phi}{2}}\right]$$

Please note :- $|E(\theta,\phi)|$ denotes the magnitude of E field intensity as a function of spherical coordinate while the angle ϕ associated with cosine is not as per spherical coordinate and defined as the angle made with array axis.

Now match with axis



z = array axis
x = either one direction principle maxima " "
y = another direction " "

Now if we consider ϕ to be the angle then the maxima can not be in x direction but in y direction. Initially this ϕ is taken as reference angle between lobe and array axis. It does not have any resemblence with (r, θ, ϕ). So to match with current figure and array coordinate system and as we assume that maxima along x axis hence ϕ may be replaced by θ for (r, θ, ϕ) coordinate system

$$\therefore |E(\theta, \phi)| = \frac{1}{N}\left[\frac{\sin \frac{N\beta d\cos\theta}{2}}{\sin \frac{\beta d\cos\theta}{2}}\right]$$

$$\therefore U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|\frac{1}{N}\left[\frac{\sin\frac{N\beta d\cos\theta}{2}}{\sin\frac{\beta d\cos\theta}{2}}\right]\right|^2 \sin\theta\, d\theta\, d\phi$$

Now for small angle $\sin x = x \rightarrow \sin\frac{\beta d\cos\theta}{2} = \frac{\beta d\cos\theta}{2}$

$$\therefore U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|\frac{1}{N}\left[\frac{\sin\frac{N\beta d\cos\theta}{2}}{\frac{\beta d\cos\theta}{2}}\right]\right|^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|\frac{\sin\frac{N\beta d\cos\theta}{2}}{\frac{N\beta d\cos\theta}{2}}\right|^2 \sin\theta\, d\theta\, d\phi$$

Let $\frac{N\beta d\cos\theta}{2} = x \Rightarrow U_0 = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi}\left|\frac{\sin x}{x}\right|^2 \sin\theta\, d\theta$

$$= \frac{1}{\cancel{4\pi}_{2}}\cdot\cancel{2\pi}\int_0^{\pi}\left|\frac{\sin x}{x}\right|^2 \sin\theta\, d\theta$$

Now as $x = \frac{Nd\beta\cos\theta}{2} \rightarrow dx = -\frac{N\beta d\sin\theta}{2}d\theta \rightarrow \sin\theta\, d\theta = \frac{-2}{N\beta d}dx$

and For $\theta = 0$ $x = \frac{N\beta d}{2}$, & For $\theta = \pi$ $x = -\frac{N\beta d}{2}$

$$\therefore U_0 = \frac{1}{2}\int_{\frac{N\beta d}{2}}^{-\frac{N\beta d}{2}}\left|\frac{\sin x}{x}\right|^2\left(-\frac{2}{N\beta d}\right)dx = -\frac{1}{N\beta d}\int_{\frac{N\beta d}{2}}^{-\frac{N\beta d}{2}}\left|\frac{\sin x}{x}\right|^2 dx$$

Now if N is very large then $\frac{N\beta d}{2} \rightarrow \infty$ & $-\frac{N\beta d}{2} \rightarrow -\infty$

$$\therefore U_0 = -\frac{1}{N\beta d}\int_{\infty}^{-\infty}\left|\frac{\sin x}{x}\right|^2 dx = \frac{1}{N\beta d}\int_{-\infty}^{\infty}\left|\frac{\sin x}{x}\right|^2 dx = \frac{1}{N\beta d}[\pi]$$

$$\therefore D = \frac{U_{max}}{\left(\frac{\pi}{N\beta d}\right)}$$

Now max Radiation intensity $U_{max} = 1$ at Boresight or at $90°$.

Hence $D = \dfrac{1}{(\pi / N\beta d)}$. Substituting $\beta = \frac{2\pi}{\lambda} \Rightarrow D = 2N\left(\frac{d}{\lambda}\right)$

$\Rightarrow D = 2\left(\frac{Nd}{\lambda}\right) = 2\left(\frac{L}{\lambda}\right)$ L = total length $(N-1)d$

Properties of End Fire array (N element)

From the previous result we obtain Broadside array. The same method can be used for end fire array with a progressive phase difference

$\theta = \dfrac{(N-1)\psi}{2} = \dfrac{(N-1)\left[\beta d \cos\phi + \alpha\right]}{2}$ | α extra phase to excitation coeff

All the element are fed in equal amplitude but progressive phase so that resultant far field along antenna axis.

$\Rightarrow$ To achieve this $\alpha = \pm\beta d \Rightarrow \psi = [\beta d\cos\phi \pm \beta d]$

So magnitude of resultant far field will be maximum if $\psi = 0$

$\therefore \beta d \cos\phi_{max} \pm \beta d = 0 \Rightarrow \cos\phi_{max} = \mp 1$

$\therefore \phi_{max} = 0°$ or $180° \longrightarrow$ along z axis.

Magnitude of major lobe $\Rightarrow$ Same as previous [Do the calculation in exam]

$|E_T| = E_0 N$

Direction of Null

Nulls are point where magnitude of resultant field become zero

$\therefore |E_T| = E_0 \left[\dfrac{\sin \frac{N\psi}{2}}{\sin \psi/2}\right] = 0 \Rightarrow \sin\frac{N\psi}{2} = 0 \rightarrow \frac{N\psi}{2} = \sin^{-1}(0) = \pm m\pi$

$m = 1, 2, 3 \ldots$

Substituting ψ $\frac{N}{2}\left(\beta d\cos\phi - \beta d\right) = \pm m\pi$ [considering $\alpha = -\beta d$ for $\phi_{max} = 0°$]

$\therefore \dfrac{Nd(\cos\phi - 1)}{\lambda} = \pm m$

Now value of $(\cos\phi - 1)$ will come either zero or negative. To cancel negative sign we assume '$-m$' as

$\dfrac{Nd(\cos\phi - 1)}{\lambda} = -m \quad \therefore \phi_{min} = \cos^{-1}\left[1 - \frac{m\lambda}{Nd}\right]$

~~Direction of Side lobe~~

Direction of Side lobe :-

Position of side lobe or subsidiary maxima can be obtained as

$\sin\frac{N\psi}{2} = \pm 1 \Rightarrow \sin\frac{N\psi}{2} = \sin\left[\pm\frac{(2m+1)\pi}{2}\right]$ $m = 0, 1, 2, \ldots$

Same as prev section

$\dfrac{N\psi}{2} = \dfrac{\pm(2m+1)\pi}{2}$

Now for $m = 0$, $\frac{N\psi}{2} = \pm\pi/2$ which gives principle maxima or major lobe. Hence the value can be avoided

$\therefore \psi = \dfrac{\pm(2m+1)\pi}{N}$ for $m = 1, 2, 3 \ldots$

Again substituting ψ $\quad \beta d\left[\cos\phi \pm 1\right] = \dfrac{\pm(2m+1)\pi}{N} \Rightarrow \dfrac{2\pi d}{\lambda}\left[\cos\phi \pm 1\right] = \dfrac{\pm(2m+1)\pi}{N}$

$\therefore$ Again $(\cos\phi - 1)$ will come either zero or negative for $\phi_{max} = 0°$

$$\therefore \cancel{\beta d}\left[\cos\phi \cancel{+} -1\right]\cdot\frac{2\pi d}{\lambda} = \frac{-(2m+1)}{N}$$

~~Now~~ again ~~$\sin x \approx x$~~ $\therefore$ ~~sin~~ $\quad \therefore \phi = \cos^{-1}\left[1 - \dfrac{(2m+1)\lambda}{2Nd}\right]$

Directivity :- $\quad D = \dfrac{U_{max}}{U_0} \rightarrow \dfrac{P_{rad}}{4\pi} = \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\int_0^{\pi} |E(\theta,\phi)|^2 \sin\theta\, d\theta\, d\phi$

$$\therefore |E(\theta,\phi)| = \frac{E_T}{|E_T|} = \frac{E_0\left[\sin\frac{N\psi}{2}\Big/\sin\psi/2\right]}{N E_0} = \frac{1}{N}\left[\frac{\sin\frac{N\beta d(\cos\phi-1)}{2}}{\sin\frac{\beta d(\cos\phi-1)}{2}}\right]$$

$$\therefore U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|\frac{1}{N}\left[\frac{\sin\frac{(N\beta d\cos\phi-1)}{2}}{\frac{\sin\beta d(\cos\phi-1)}{2}}\right]\right|^2 \sin\theta\, d\theta\, d\phi$$

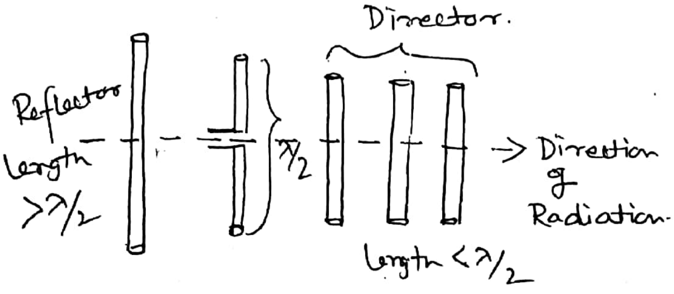
Again $\sin x \approx x$ $\quad \sin\dfrac{\beta d\cos\phi-1)}{2} = \dfrac{\beta d(\cos\phi-1)}{2}$.

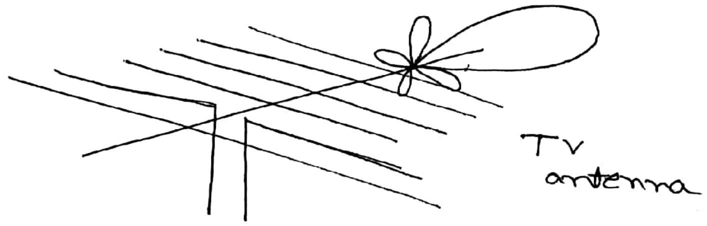
Same procedure as prev [do yourself]

$$D = \frac{U_{max}}{\left[\frac{\pi}{2N\beta d}\right]} = \frac{1}{\left[\frac{\pi}{2N\beta d}\right]} \Rightarrow \beta = \frac{2\pi}{\lambda} \therefore D = 4\pi\left(\frac{d}{\lambda}\right) = 4\left(\frac{Nd}{\lambda}\right) = 4\left(\frac{L}{\lambda}\right)$$

YAGI-UDA Antenna (parasitic End Fire array) HF (3–30 MHz), VHF (30–300 MHz) UHF (300–3000 MHz)

Yagi Uda antenna is a high directive antenna first proposed by Prof Uda and described by Prof Yagi in 1930. It is an array of parasitic element. All the elements are dipole in nature and are parallel to feed dipole element. In general the structure consists of a feed element (Driven element of $\lambda/2$ dipole), a reflector, made of metallic rod of somewhat length greater than driven element and ~~some~~ more than one director, made of metallic rod of length less than resonant driven element length. Longer and shorter element are kept opposite side of halfwave driven element.

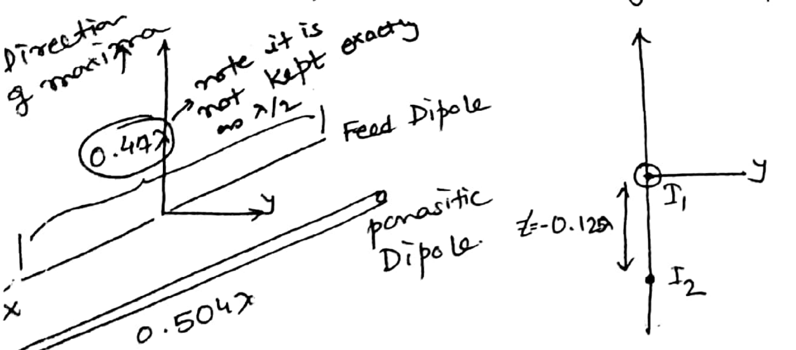




Parasitic elements are reflector and director. It is an element in which current is induced by the field in the nearby element. It does not require any physical connection with transmission line to receive power. Thus it forms a magnetically coupled array of parasitic element. Length and separation between elements affects the directional pattern of the array. Thus better direction can be achieved by properly trimming the physical length and separation. If the parasitic element is more than the resonant length ($>\lambda/2$) then it is inductive in nature and acts as reflector and similarly if the length is lesser than half wavelength ($<\lambda/2$), then it is capacitive in nature and act as director. In general the separation between the elements is kept between 0.1λ to 0.15λ. Generally the gain is of 9 to 12 dB. It depends upon no. of director but you can not increase director more than 9 − 10 as more distant director, it receives very weak power through coupling. The spacing between element is chosen to keep 2–3% BW required for TV reception. (57)

Now as reflector has inductive in nature hence phase of current lags the induced voltage. With proper spacing subject to driven element, field from reflector adds up to field of driven element in the direction of driven element. Similarly director being length less than resonant length, is capacitive in nature. Hence phase of current leads the induced voltage. Therefore, the field from director adds up to the fields of driven element in the direction away from driven element.

Analysis :- We will consider two case with one dipole and one non exited dipole element. They are parasitically coupled to each other.



Feed dipole is placed at origin. Second dipole, terminal short circuited is brought near to the driven dipole and kept ‖ to it. Hence a current is established on second dipole due to EM induction. Let the current on the driven element is 1A, induced current on the parasitic element be $0.7\angle 140^\circ$.

So for two element array $AF = I_1 e^{jkz_1'\cos\theta} + I_2 e^{jkz_2'\cos\theta}$

$I_1 = 1\,A$, $I_2 = 0.7\angle 140^\circ$, $z_1' = 0$ and $z_2' = -.125\lambda$.

$\therefore\ AF = 1 + 0.7 e^{j(7\pi/9 - \pi/4)\cos\theta}$

So the magnitude of AF along $\theta = 0^\circ$ is $|AF| = |0.94 - j0.7| = 1.172$ and at $\theta = 180^\circ$ is $|AF| = |1.303 - j0.06| = .309$. It proves the antenna radiates more power in $+z$ direction compared to $-z$ direction. So it seems that the parasitic element is reflecting the incident field on it. So we can assume it to be a reflector.

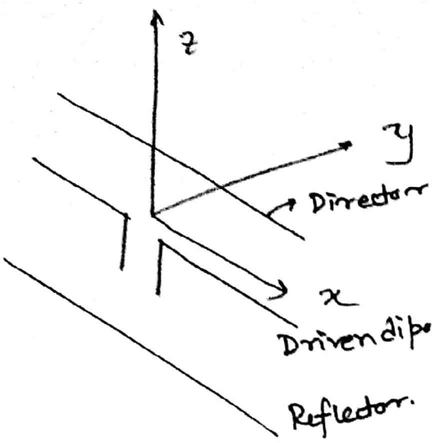
If ~~driven~~ parasitic dipole is 0.47λ and ~~parasitic~~ Driven dipole is 0.504λ then current on parasitic element leads the current on driven element.

Now for the same 2 element case $I_1 = 1A$ and $I_2 = 0.7\angle -140^\circ$ A then

$AF = 1 + 0.7 e^{j\left(-\frac{7\pi}{9} - \pi/4\right)\cos\theta}$.

$\Rightarrow$ For $\theta = 0^\circ$ $|AF| = .309$ & For $\theta = 180^\circ$ $|AF| = 1.172$. Thus the dipole field scattered from the parasitic element add along the direction of parasitic element from dipole. Thus the parasitic element is called director.

Now consider a three element array case



One driven element of resonant dipole and two parasitic element (Reflector & Director).

length of Driven element 0.47 λ

" " Director 0.442 λ

" " Reflector 0.482 λ

optimum spacing Reflector to Driven element 0.15λ – 0.25λ

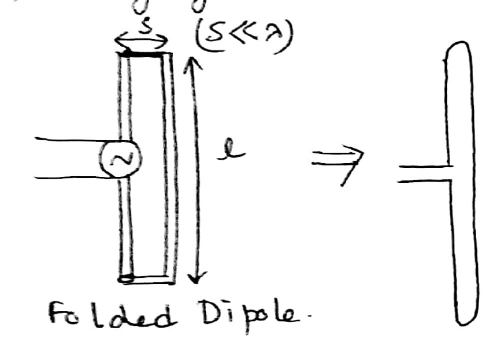
" " Driven element to Director 0.2λ – 0.35λ.

Directivity of Yagi Uda can be increased if more Director can be added with a interval of 0.18λ to 0.3λ

In general only one Reflector is used because the field of the antenna just behind the reflector is small. Hence a second reflector will not be effective since very small current is induced.

Feed problem :- In general to achieve a good directional property with 2-3% BW Half wave Dipole of $Z_{in} = 73\Omega + j42.5$ are used. Now to feed this dipole the impedance of the feed line be of the order of 75Ω. But for a TV receptia, the twin wire line approximately require 300Ω of impedance. Thus, there are some mismatch problems between single Dipole and Feed line.

In order to have good matching, variation of single Dipole element has been used. We use folding of structure in such a way that it looks like two array of intermidiate gab $\ll \lambda$. It acts like a step up transformer.

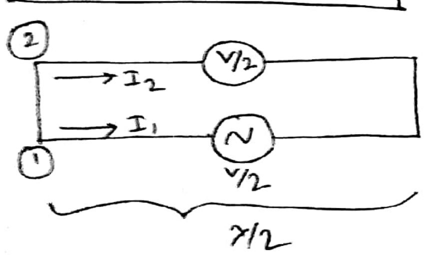


Folded Dipole.

Folded dipole offers high input impedance compared to half wave dipole due to current distribution and input impedance depends on the number of arms in the folded dipole. If the number of arms in the dipole is m and all the arms are identical i.e radius is same then the $\boxed{Z_{in} = m^2 R_r}$

So For a two arm folded dipole with length λ/2, the input impedance comes out as 292Ω. The input impedance of Folded dipole can be varied over a range by changing the separation and radius of dipole.

$\boxed{\text{Proof } Z_{in} = m^2 R_r}$



Input impedance of Folded dipole can be derived by using the equivalent circuit as shown in fig left. Let V voltage applied to Folded dipole and it is divided equally in each arm. So From the self and mutual impedance

$$V/2 = Z_{11} I_1 + Z_{12} I_2$$

Z_{11} = self impedance of Dipole 1, Z_{12} = mutual impedance between 1 and 2

Now if $I_1 = I_2$ then $V/2 = (Z_{11} + Z_{12}) I$

Now for $S \ll \lambda$ we can consider $Z_{11} \approx Z_{12}$

$\therefore V/2 = (2Z_{11}) I \quad \therefore V = (4Z_{11}) I, \quad \boxed{V/I = 4Z_{11}}$

So as if it behaves like a step up transformer

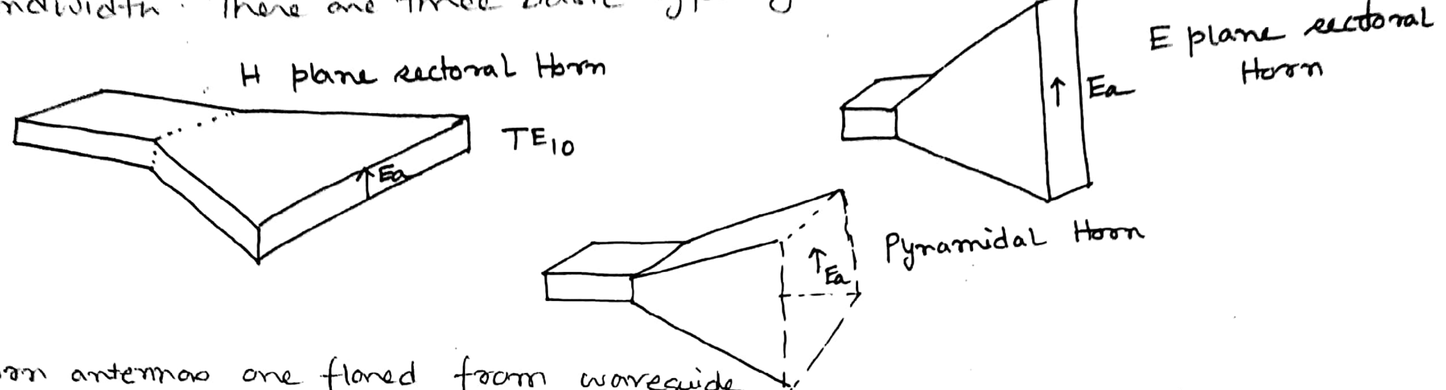
So $Z_{in}(FD) = \frac{V}{I} = 4Z_{11}$. For self impedance of Dipole $Z_{11} = 73\Omega$

$\therefore Z_{in} = 4 \times 73\Omega = 292\Omega = 2^2 \times 73\Omega \Rightarrow Z_{in} = m^2 \times 73\Omega\, R_r$

Horn Antenna (Impedance matching transformer)

Horn antenna classified under aperture antenna. Aperture antenna emit EM wave through an opening (aperture). The pupil of human eye is a typical aperture receiver for optical radiation. At radio and microwave frequency horn, waveguide aperture, reflectors, are some of aperture antenna. Aperture antennas are commonly used at UHF and above where their size is reasonable.

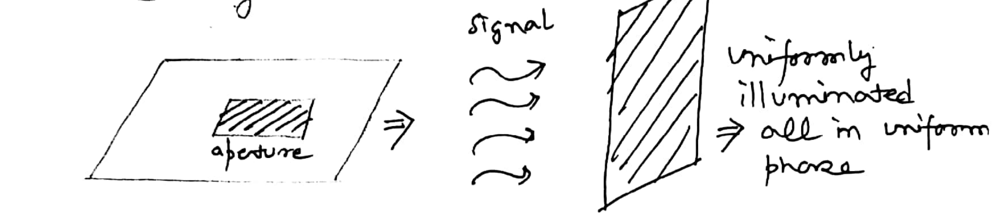
Rectangular Horn antenna :- Horn antennas are popular in the microwave bands (above 1GHz). Horns provide high gain, low VSWR and relatively wide bandwidth. There are three basic types of horn.

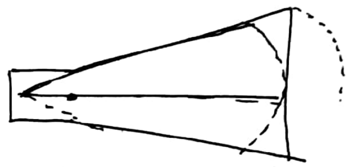


Horn antennas are flared from waveguide opening $(a \times b)$ to a larger dimension. This provides better impedance match in a broad frequency band. Rectangular horns are named by rectangular feed. If the feed is cylindrical waveguide, the antenna is called conical horn.

What is the new thing for horn antenna compared to simple flat aperture (e.g. rectangular aperture on a metal sheet)

See the Fig

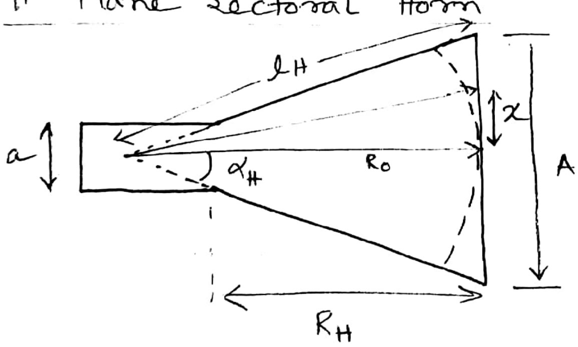




See the opening has not received signal from feed uniformly. See the phase diff between center and edge.

Phase error.

H Plane Sectoral Horn



H Plane X-Z cut of the horn

$$l_H^2 = R_0^2 + \left(\frac{A}{2}\right)^2$$

$$\alpha_H = \tan^{-1}\left(\frac{A}{2R_0}\right)$$

$$R_H = (A-a)\sqrt{\left(\frac{l_H}{A}\right)^2 - \frac{1}{4}}$$

The tangential field arriving at the input of horn is composed of transverse field component of waveguide dominant mode TE_{10}

$$E_y(x) = E_0 \cos\left(\frac{\pi}{a}x\right)e^{-j\beta z}$$

$$H_x(x) = -E_y(x)/Z_g$$

where $Z_g = \dfrac{\eta}{\sqrt{1-\left(\frac{\lambda_0}{2a}\right)^2}}$

& $\beta = \beta_0\sqrt{1-\left(\frac{\lambda_0}{2a}\right)^2}$

Again $\beta_0 = \omega\sqrt{\mu\epsilon} = \dfrac{2\pi}{\lambda_0}$

So it means that field illuminating the aperture of horn is essentially a spatial expanded version of waveguide field. Wave impedance gradually approaches to η.

The main problem of horn analysis compared to waveguide is that the wave arriving at horn aperture are not in phase due to different path length from the horn apex.

The aperture phase variation is $e^{-j\beta(R-R_0)}$

Since the aperture is not flared in y direction, the phase is uniform along y.

$\therefore$ So for an arbitrary point

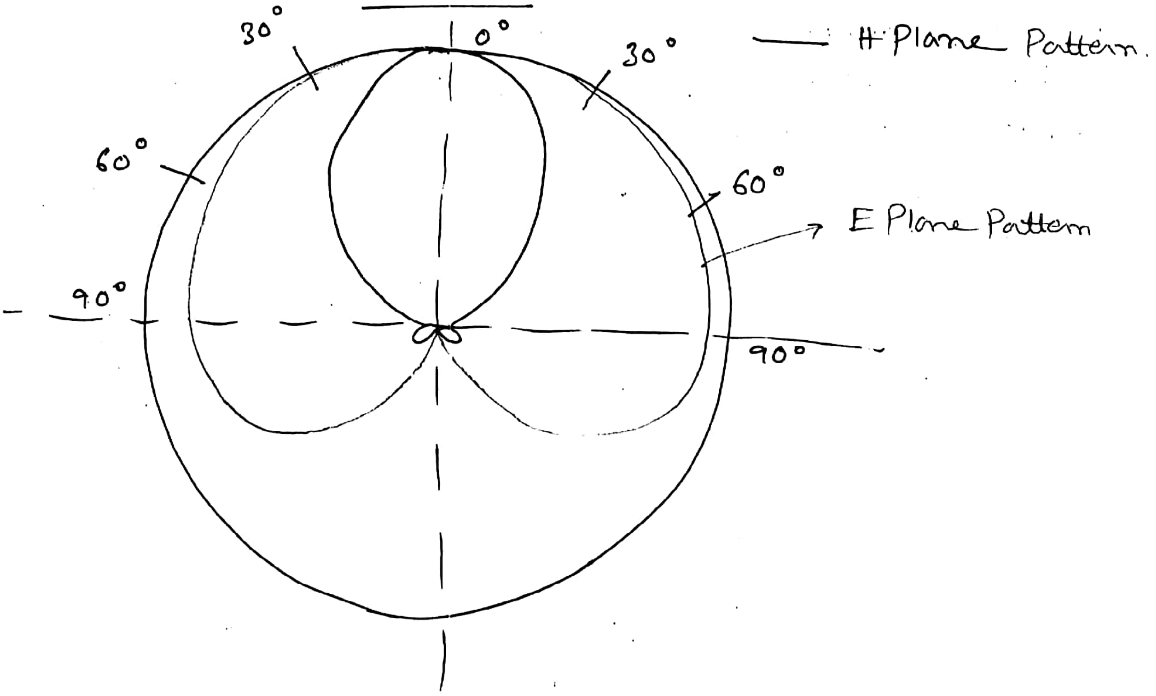
$$R = \sqrt{R_0^2 + x^2} = R_0\sqrt{1+\left(\frac{x}{R_0}\right)^2} \approx R_0\left[1+\frac{1}{2}\left(\frac{x}{R_0}\right)^2\right]$$

Now this holds if $x \ll R_0$ or $A/2 \ll R_0$ then

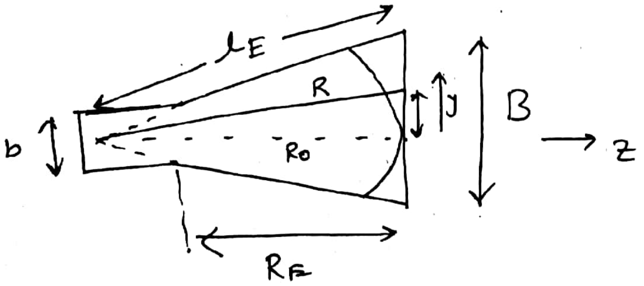
$$R - R_0 \approx \frac{1}{2}\frac{x^2}{R_0}$$

$$\therefore \quad F_{ay}(x) = E_0 \cos\left(\frac{\pi}{A}x\right) e^{-j\frac{\beta}{2R_0}x^2}$$

Pattern



E Plane Sectoral horn.



As previous

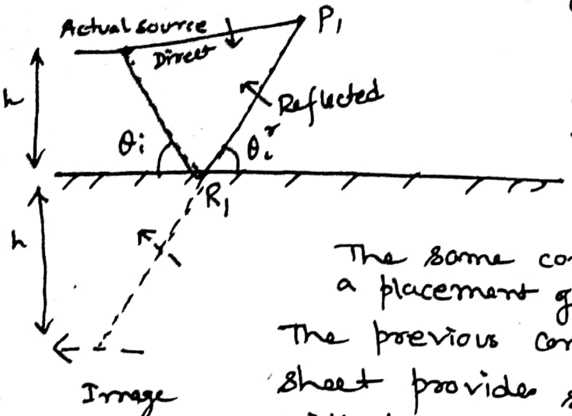
$$E_{ay} = E_0 \cos\left(\frac{\pi}{a}x\right) e^{-j\frac{\beta}{2R_0}y^2}$$

$$R - R_0 \approx \frac{1}{2}\frac{y^2}{R_0}$$

Summary :- Horn antenna is a simple open ended waveguide where wave impedance of waveguide has been matched gradually to free space wave impedance. At this moment minimum signal reflects back to feed and max of energy transmitted to free space. Otherwise max signal will reflect back due to impedance mismatch.

Reflector Antenna

Suppose a linear dipole (Horizontaly polarized) is placed horizontally relative to the infinite electric ground plane. For analysing this kind of geometry we need image theory.

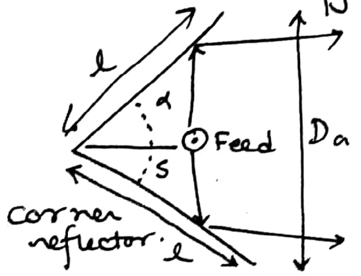


Directed and Refleted wave can be approximated from ground reflected wave analysis (Propagation)

Reflection coefficient $= -1$

The same concept of reflection from a conducting surface due to a placement of simple dipole can be used as reflector antenna. The previous concept of simple feed (Dipole) infront of conducting sheet provide simple reflection. That means ~~the~~ a lot of energy will be lost due to uncollimated reflection.

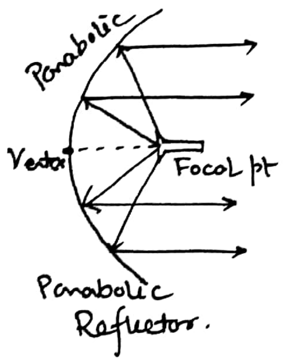
If we want to channelise the total reflection to a single direction, then we need to modify the reflecting surface geometry.



To have better collimation of energy in a specified direction, the shape of plane reflector has been changed so that no side reflection is happen. For this, two plane reflectors are joined so as to form a corner reflector. this reflector is used as passive target for communication application. It will return signal exactly in the same direction as it received it when it included angle is 90°. In general α is 90°. In general S and α provides the radiation efficiency and gain of antenna.

In genral the aperture of corner reflector Da is chosen as $\lambda < D_a < 2\lambda$.

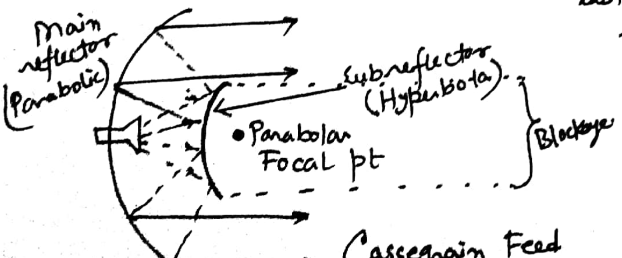
$l \approx 2s$.



Overall radiation characteristic of a reflector can be enhanced with the left side structural modification over corner reflector. It has observed and physically proved from geometrical optics that if a beam of $\parallel^r$ rays incident upon a reflector of parabolic shape, the radiated energy converge to a spot called focal pt. Hence if a feed is placed at the focal point then rays reflected from reflector will provide $\parallel^r$ beam due to principle of reciprocity. The symmetrical point on parabolic surface is known as vertex. Since the Feed is placed at the focal pt of parabola, this configuration is usually refered as front fed.

The ~~dista~~ disadvantage of front fed arrangement is that the transmission line from the feed must usually be long enough to reach the actual tx or Rx system which need to be placed very close to reflector to minimise the loss. Some time; the system are placed on the focal feed point with heavy and bulky equipment. This provide undesirable aperture blockage.

Above arrangement has been modified so that feed ~~can no~~ need not to be placed at the focal point. The process is called <u>Cassegrain Feed</u>. Cassegain, a famous astronomer showed that incident $\parallel^r$ rays can be focused to a point by utilizing two reflectors. Here the primary reflector must be a parabola and secondary reflector is a hyperbola. Feed placed along the axis of the parabola usually at vertex. This scheme was used for construction of optical telescope. Here the rays emanate from feed

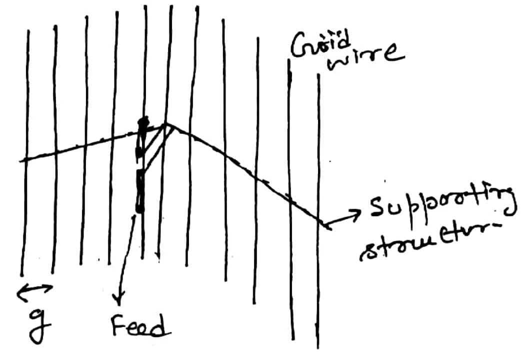


Cassegrain Feed

illuminate the subreflector and are reflected by it in the direction of primary radiator as if they originated at the focal point of parabola. the rays are then reflected by primary reflector and are converted to $\parallel^r$ rays.

Only problem of this kind reflector is the diffraction occur at the edges of the reflector. Both simple focal pt feed and Cassegrain feed suffer from aperture blockage due to placement of reflector / Feed Infront of collimating direction. The problem can be solved by placing offset Feed in reflector antenna structure.

★ In some application, specially when wavelength is large compared to tolerable physical dimension, the surface of ordinary corner reflector (solid metal body) are modified to grid wires rather than solid sheet. One reason for doing that is to reduce wind resistance and overall system weight. The spacing (g) between grid wire is made small fraction of λ ($g \leq \lambda/10$). For wires $\parallel^r$ to dipole, the reflectivity of grid wire surface is as good as solid surface



★ Assignment : Do it yourself $\rightarrow$ Take a $\lambda/2$ Dipole. Take the Farfield E field. Then ~~add~~ take a metal screen infront of Dipole. Choose the Image of Dipole. ~~Rept~~ Remove the screen. Then you have two Dipole at two different distance w.r.t origin. Calculate the overall field E for E_{ac} & E_{image}. Then plot the pattern for distance of $\lambda/4$ ~~from~~ between screen and actual dipole.

Broad band antenna

Some times antenna are required to operate at a certain range of frequency. The antennas with wide bandwidth are called as broad band antenna. The term broadband is a relative measurement of bandwidth and varies with application. Bandwidth is computed in one of two ways: Let f_U and f_L be upper and lower frequencies of operation for which satisfactory performance is obtained. The center is denoted as f_C.

So % of BW subject to % of center frequency is $\dfrac{f_U - f_L}{f_C} \times 100$.

Again BW sometimes can be referred as f_U/f_L

In general BW of narrow band antenna is usually refrared as % of BW whereas for wide band antenna simple ratio are used.

In general resonant antennas have small bandwidth (8 – 16 %). On the other hand travelling wave antenna operates for a wide band. Traditional definition suggest that if the impedance and pattern of an antenna do not change significantly over about an octave ($f_U/f_L = 2$) or more then it is classified as broad band antenna.

Spiral antenna :- From the def^n of broadband antenna, the design can be further extended where pattern and impedance bw of antenna remains unchanged for a wide range of frequency (say 10:1). Such type of antennas are called as frequency independent antenna.

★ One important property of frequency independent antenna should be noted. A complementary Consider a metal antenna with the input impedance Z_{metal}. A complementary structure can be formed which is an antenna with air replacing metal and metal replacing air. Let the impedance of this complementary antenna is Z_{air}. An example can be a dipole of metal rod and it's complementary antenna is a slot / aperture of same length of dipole on a metal sheet

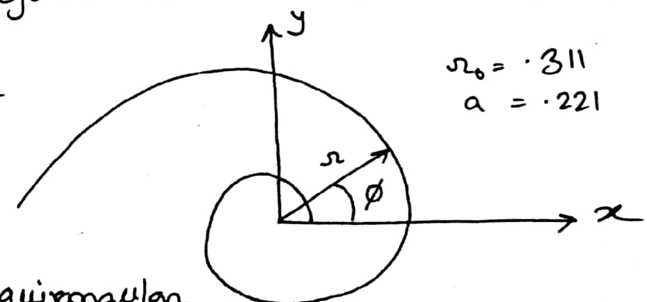
Then it can be proved that $Z_{air}\, Z_{metal} = \eta^2/4$

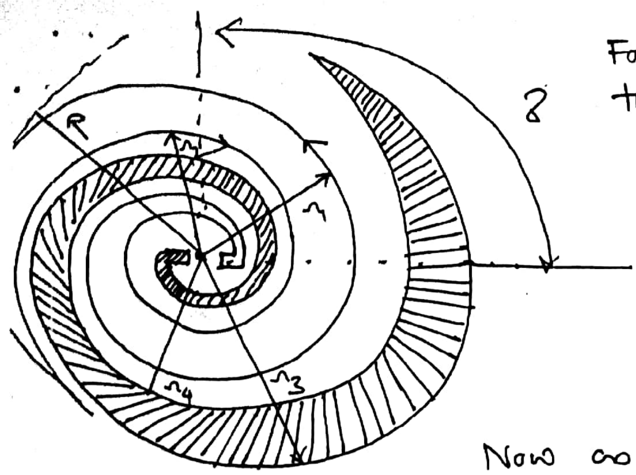
If an antenna and complementary are same (idealy) they are called self complementary and then $Z_{air} = Z_{metal} = \eta/2 = 188.5\,\Omega$

This relationship is frequency independent and important consideration for design of frequency independent antenna.

⇒ Let an equiangular spiral curve. In cylindrical coordinate the distance of the curve from origin is $r = r_0 e^{a\phi}$

r_0 = radius for $\phi = 0$ and a is a constant giving the flare rate of spiral. Now using this spiral curve equiangular geometry can be used to produce the angular antenna which is referred as planar equiangular spiral antenna.





Four edges of metal follows the equation for their curves

So $r_1 = r_0 e^{a\phi}$

edge no 2 has same spiral but rotated through the angle γ so $r_2 = r_0 e^{a(\phi-\gamma)}$

Again $r_3 = r_0 e^{a(\phi-\pi)}$
and $r_4 = r_0 e^{a(\phi-\pi-\gamma)}$ } Complementary.

Now as they are identical so $\Rightarrow$ self complementary

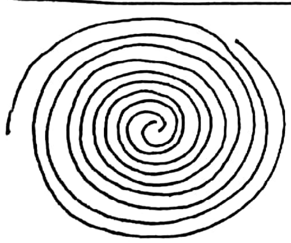
So $\gamma = \pi/2$

Typical impedance as mentioned previously $188.5 + j0\ \Omega$.

☆ Field pattern $\Rightarrow$ bi directional with two wide beams broadside to plane of antenna.

Polarization $\Rightarrow$ close to circular over wide angles.

Archimedean spiral :-



constructed on printed domain (PCB)

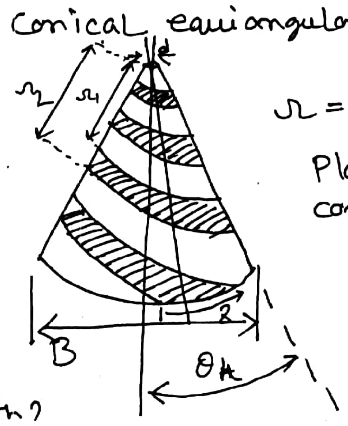
The equation of two spiral $r = r_0\phi$ and $r_0 = (\phi - \pi)$

Properties are similar to equiangular planar spiral.

If you want to remove one beam then cavity backed antenna to be used by placing a circularly cylindrical cavity on one side of spiral. Commercial antenna have 90° HPBW over 10:1 Bandwidth.

Non Planar Archimedean spiral :- Non planar form can be obtained by using a conformal structure where a conical surface is used and use the same planar / wire printed on conical surface

Conical equiangular spiral antenna:



$r = e^{(a\sin\theta_h)\phi}$

Planar spiral is a special case of conical spiral where $\theta_h = 90°$.

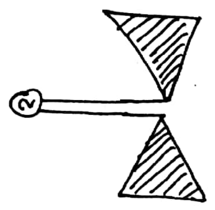
How to determine Bandwidth?

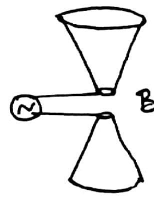
f_U the upper operating frequency depends on d with special relationship $d = \frac{\lambda_U}{4}$

f_L the lower band edge is determined by Base diameter B and the relationship is $B = 3\lambda_L/8$. $\theta_H \approx < 15°$

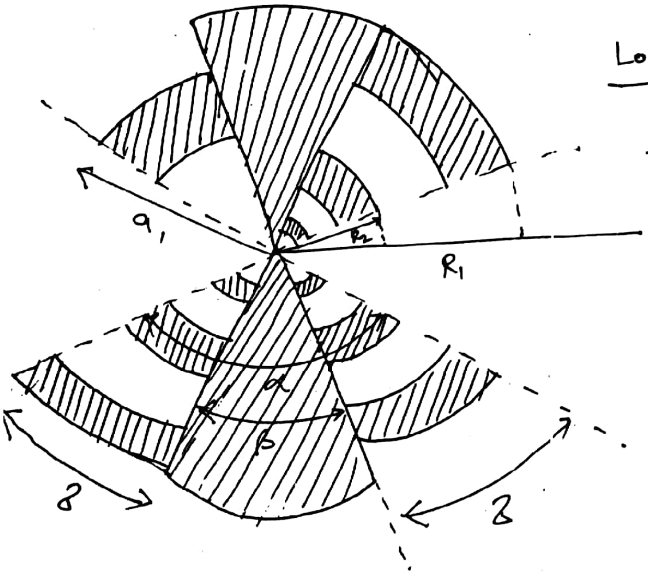
Log Periodic Antenna :- In spiral antenna the primary emphasis is on angle. One problem of spiral antenna is complex structure involving curve generation. Generally if the construction is involved simple circular or straight wire then design can be simplified.

Now the most commonly used such kind of antenna is Bowtie antenna. It is a planar version of biconical antenna.



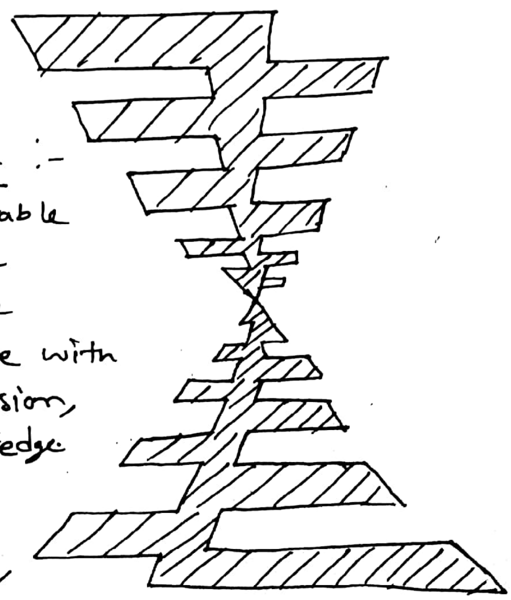


It has bidirectional pattern with broad main beam $\perp^r$ to the plane of antenna. But it is linearly polarized. Generally Bowtie antenna is used as receiving antenna for UHF TV channels. Now the main problem of this kind of antenna is currents are abruptly terminated at the ends of structure, thus antenna has <u>limited Bandwidth</u>. Now a simple Bowtie antenna can be modified to <u>Broad Band antenna</u>, ~~curr~~ where currents will then die off more rapidly with distance from feed point. Thus this kind of periodic teeth implemented ~~are~~ is called <u>log periodic antenna</u>.



<u>Log Periodic toothed Planar antenna</u>

It is similar to Bowtie antenna but only ~~teeth~~ teeth are added. The teeth act to disturb the currents which would flow if the antenna were of simple Bowtie type. Current flow out along the teeth and except at the frequency limits are not significant at the ends of [redacted] antenna.



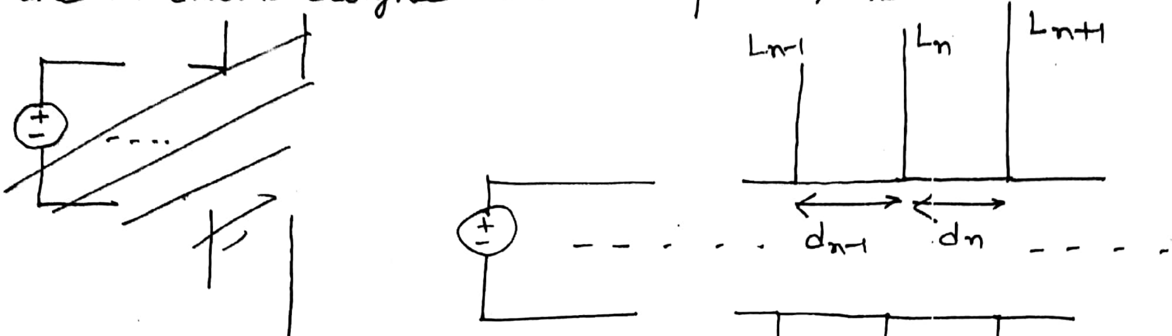
<u>Log Periodic toothed trapezoidal antenna</u> :-

From the construction point of view it is desirable if the toothed antennas could be made with ~~and~~ straight edges. This simplification of the structure turns out to be little consequence with antenna performance. For trapezoidal version, bending of planar version changes into wedge ~~creating~~

Why such structure is known as log. periodic?

In general from Bow tie antenna we have seen that wideband antennas are often defined by angles instead of length. i.e why they are frequency independent. As an alternative to this, if the antennas have self similar structure so that properties at some frequency $f_2 = K \times f_1$ is identical at first frequency f_1. $K > 1$

Let the antenna is designed at some frequency f_n. Now look at the figure.



$$\frac{L_{n+1}}{L_n} = \frac{d_{n+1}}{d_n} = K$$

Wire lengths are $L_{(n-1)}$, L_n and $L_{(n+1)}$ separated by distance $d_{(n-1)}$, d_n and $d_{(n+1)}$

We need the ratio of successive element $L_{(n+1)}/L_n$ be equal to some constant K and that the distance between elements $d_{(n+1)}/d_n$ is also K.

So from figure antenna radiates well at frequency f_n (Primary to L_n). Then the antenna must also radiate at $f_{(n+1)}$ & $f_{(n-1)}$ as the antenna structure is electrically same.

$$\Rightarrow \quad \lambda_n = \frac{c}{f_n} \rightarrow \lambda_{n-1} = \frac{c}{f_{n-1}} \longrightarrow \lambda_{n+1} = \frac{c}{f_{n+1}}$$

$$\Rightarrow \quad \frac{f_n}{K^2}, \quad \frac{f_n}{K}, \quad f_n, \quad K f_n, \quad K^2 f_n$$

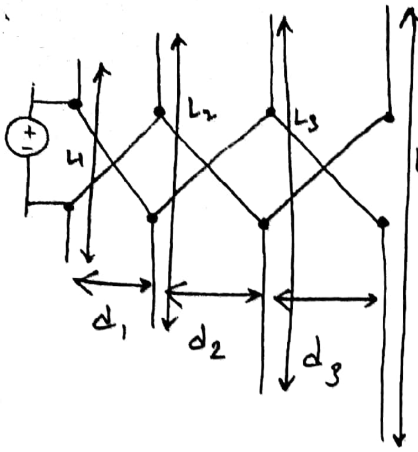
Means antenna radiates at frequency f_n and also all the other frequency which are constant multiple of f_n

Now if f_n has a certain Band width and f_{n+1} has also certain BW with a special overlap with f_n then we have a good radiation efficiency between f_n and f_{n+1}. Thus wide band nature comes into play.

$$\text{Now} \quad \log \frac{f_{n+1}}{f_n} = \log(K) = \log \frac{d_{n+1}}{d_n}$$

Mathematically due to the properties of logarithms, if all the element grows by a constant multiple then the ratio of logarithm will be constant

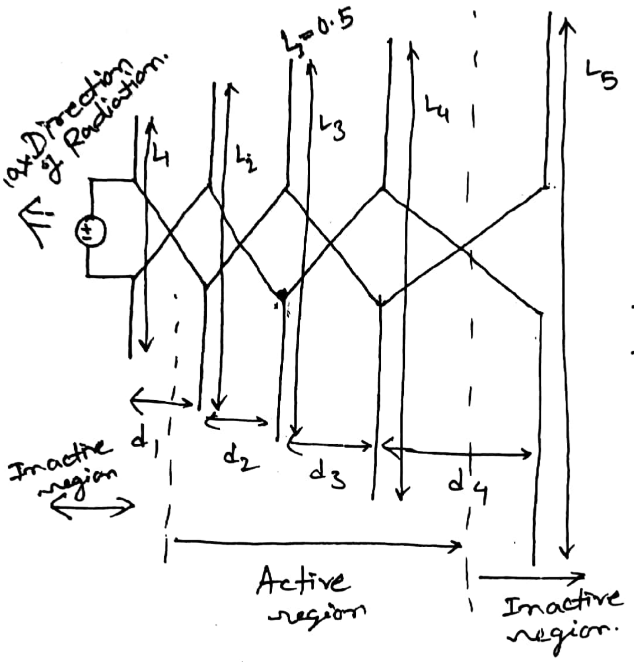
Now the properties of log periodic nature can also be incorporated to log periodic dipole array (LPDA) where individual dipole element of increasing length have to be fed with 180° out of phase connection between alternative successive dipole element. Be careful of this geometry as it is very much similar with Yagi Uda antenna.



$$\frac{L_{n+1}}{L_n} = \frac{d_{n+1}}{d_n} = K$$

Let $K = 1.25$. this means that each dipole is 25% longer than one to the left of it and the separation (d) between each dipole is also increased by 25%. More over, array is arranged in such a way that each element is fed out of phase to the element on either side.

Let for a Five element antenna discuss the regional variation.



This antenna is often characterized by active and passive zone. Say we need to mechanice the radiation at $f = 300$ MHz, then the bulk of the radiation from the antenna will come out from the dipole with lengths near half a wavelength at 300 MHz. $(L = 0.5)$

the elements ~~....~~ that are too short will be too capacitive to radiate and the element much Longer than $\lambda/2$ will also not radiate well.

Zig zag Fed so that signal phase is correct